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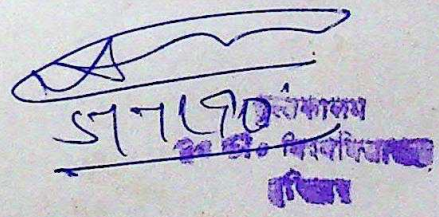

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# ON A BEST APPROXIMATION THEOREM

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*( Received : June 15, 1988 ; Revised : October 30, 1988 )*

## ABSTRACT

In this paper a Theorem on Best Approximation is proved . It extends results due to Ky Fan, Hadzic, and Sehgal, Singh and Smithson .

A well known theorem due to Ky Fan [2] has been of great importance in nonlinear analysis, minimax theory and approximation theory . This theorem has been further extended by several mathematicians and applied in various areas including game theory and the theory of Economics .

In this paper we state a best approximation theorem in a topological vector space which contains as particular cases the above, an other result recently given by Ky Fan [3], and its generalizations due

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to Sehgal, Singh and Smithson [10] , and Hadzic [4] . This theorem is an useful tool to obtain nearest point theorems, coincidence theorems and fixed point theorems .

We need the following definition :

Let $E$ be a vector space and $p$ a seminorm on $E$, $X$ a non-empty convex subset of $E$. A map $g : X \to E$ is said to be almost $p$-affine if for all $x, y \in X$ $u \in E$ and $r$, $0 < r < 1$ satisfies

(1) $p(g(rx + (1-r)y) - u) \leqslant rp(gx-u) + (1-r)p(gy-u)$ .

Now we state our main theorem :

**Theorem 1.**

Let $(E,T)$ be a topological vector space, $X$ a non-empty convex subset of $E$, $S$ a non-empty compact convex subset of $X$ and $K$ a compact subset of $X$. Let $p$ be a seminorm on $E$, $f : (X, T) \to (E, p)$ be a continuous map and $g : X \to E$ be an almost $p$-affine map such that $p\,g$ is lower semicontinuous on $(X, T)$ . If $f$ satisfies the condition :

(2) for every $y$ in $X/K$ there exists an $x \in S$ satisfying $p(gx-fy) < p(gy-fy)$,

then there exists a $u \in K$ such that

(3) $(pgu-fu) = \min \{ p(gx-fu) : x \in X \}$ .

If moreover, g is onto $X$, then we have

(4) $p(gu-fu) = min(p(x-fu) : x \in X\}$.

**Proof**

The proof is similar to that of Ky Fan's Theorem 7 in [3]. In the proof we need the following theorem due to Allen [1] .

**Theorem (A) :**

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Let $X$ be a non-empty convex subset of a topological vector space $E$. Let $f : X \times X \to R$ be such that

(i) for each fixed $x \in X$, $f(x, y)$ is a l. s. c. function of $y$ on $X$.

(ii) for each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on $X$,

(iii) $f(x, x) \leqslant 0$ for ail $x \in X$.

**Furthermore,**

(iv) let $X_0$ be a compact, convex non-empty subset of $X$ such that the set $(y \in X : f(x, y) \leqslant 0$ for all $x \in X_0)$ is compact.

Then there exists a $y_0 \in X$ such that $f(x, y_0) \leqslant 0$ for all $x \in X$.
Define $G : X \times X \to R$ by

$$G(x, y) = p(gy - fy) - p(gx - fy).$$

For every $x \in X$, $G$ is a lower semicontinuous function of $y$ and $G(x, x) = 0$.

Now we show that $G(x, y)$ is a quasi-concave function of $x$ for every $y \in X$.

Indeed, if $G(x_1, y) > t$ and $G(x_2, y) > t$, then because of (1)
$G(rx_1 + (1-r)x_2, y) \geqslant rG(x_1, y) + (1-r)G(x_2, y) > t$.
Finally, we show that the set

$$A = \{ y \in X : G(x, y) \leqslant 0 \text{ for all } x \in S \} \text{ is compact.}$$

Indeed, if $A = \phi$ then it ts obvious. If $A \neq \phi$, then since $G$ is a lower semicontinuous function of $y$ for all $x \in S$, A is a closed set; because $y \in A$ if and only if $y \in X$ and $p(gy - fy) \leqslant p(gx - fy)$ for all $x \in S$, A is a subset of $K$ (by (2)). Therefore A is compact.

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Thus $G$ satisfies all the conditions of Theorem (A), therefore there exists a $u \in X$ such that $G(x, u) \leqslant 0$ for all $x \in X$. Moreover, $u \in A \subset K$, hence $u \in K$ and satisfies (3).

If, in particular, $X$ is a compact set, we may assume in Theorem 1, $X = S = K$. The Condition (2) is obviously satisfied and we obtain as a corollary the following :

**Theorem 2.**

Let $(E, T)$ be a topological vector space, $K \subset E$ a non-empty convex compact set and $p$ a seminorm on $E$. Let $f : (K, T) \to (E, p)$ be a continuous map and $g : K \to E$ be an almost $p$-affine map such that $p \cdot g$ is lower semicontinuous on $(K, T)$. Then there exists a $y \in K$ such that $p(fy - gy) = \min \{p(fy - gx) : x \in K\}$.

If, moreover, $g$ is onto $K$, we have

$p(fy - gy) = \min \{p(fy - x) : x \in K\}$.

Now we derive some consequences of Theorems 1 and 2 in order to obtain various results as corollaries in Approximation Theory.

1. Suppose $(E, T)$ is a normed linear space and $p(x) = \| x \|$. Then Theorem 1 contoins a result of Hadzic ([4], Theorem 4) and Theorem 2 gives a result due to Prolla [8], In case $g(x) = x$, then we get from Theorem 1 a well-known theorem of Ky Fad ([3] Theorem 7). and Theorem 2 yields another famous theorem due to Ky Fan ([2], Theorem 2).

2. Suppose that $E$ is a normed linear space and $T$ is the weak topology of $E$. If we assume that $p(x) = \| x \|$, and $g(x) = x$ we obtain from Theorem 1 a result due to Sehgal, Singh and Smithson

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( [ 10 ], Theorem 3 ), and from Theorem 2 a result due to Kapoor ( [ 7 ], Theorem 2). ( Remember that $x \to \| x \|$ is lower semicontinuous on $(E, T)$ ) .

3. Analogous results hold if $E$ is a dual normed space and $T$ is the weak * topology of $E$. ( Indeed, $x \to \| x \|$ is lower semicontinuous also in the weak * topology of $E$ ) .

4. Suppose that $(E, T)$ is a locally convex topological vector space and $P$ is the family of seminorms which generates $T$. Since any seminorm $p \in P$ is continuons on $(E, T)$ and $f : (X, T) \to (E, T)$ continuous obviously imply that $f : (X, T) \to (E, p)$ is continuous, Theorem 2 contains Theorem 1 of Ky Fan [2] and a result due to Sehgal and Singh [9] .

5. Suppose that $T$ is the weak topology of a locally convex topological vector space $(E, \tau)$ and $P$ is the family of seminorms which generates $\tau$. Then since every $p \in P$ is lower femicontinuons on $(E\ T)$ and $f : (X, T) \to (E, \tau)$ continuous imply that $f : (X, T) \to (E, p)$ is continuous, so Theorem 1 contains Theorem 1 of Sehgal, Singh and Smithson [10] .

6. Finally, Theorem 1 obviously contains Theorem 4 of Hadzic [ 4, p. 34 ] . Theorems 1 and 2 also allow us to obtain coincidence results .

Let us recall the inwardness condition . This condition is weaker than the condition $x \in \partial X \Rightarrow fx \in X$, ( $\partial X$ stands for boundary of $X$ ) and is widely used in order to give fixed point theorems for mappings $f : X \to E$ ( $X \subset E$ ) .

For $x \in X$ the inward and outward sets of $X$ at $x$ are defined by

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$I_X(x) = \{ x + r(y-x) \in E : y \in X, r > 0 \}$ .

$O_X(x) = \{ x - r(p - x) \in E : y \in X, r > 0 \}$ .

If $x$ is an internal point then $I_X(x) = O_X(x) = E$ .

If $f : X \to E$ and $fx \in I_X(x)$ ( $O_X(x)$ ) for each $x \in X$ then $f$ is said to satisfy inward (outward) condition .

In case $X$ is a convex set this condition may be given as follows ( see Massa [7] ) .

A map $f : X \to E$ is an inward map if

(5) $x \in X \Rightarrow \{ y \in E : y = rx + (1-r) fx, 0 \leqslant r < 1 \} \cap X \neq \phi$ .

In the case $X$ is a convex set and $g$ is onto $X$, we introduce the following more general definition :

A map $f : X \to E$ is said to be $g$-inward if

(6) $x \in X \Rightarrow \{ y \in E : y = rgx + (1-r) fx, 0 < r < 1 \} \cap X \neq \phi$ .

The following theorems hold :

**Theorem 3 ( Theorem 4 ) .**

Suppose that the hypotheses of Theorem 1 ( Theorem 2 ) are satisfied and $g$ is onto $X$. If $f$ is a $g$-inward map, then there exists a point $u \in K$ such that :

(7) $p(gu-fu) = 0$ .

If moreover, $x \in (g-f)(K)$ and $p(x)=0$ imply that $x=0$, then we have $fu = gu$ .

**Proof :**

Let us consider a point $u$ which satisfies (4). If $fu \in X$, then $u$ satisfies

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(7). If $fu \notin X$, let $r, 0 < r < 1$ such that $rgu + (1\text{-}r)fu \in X$ Then (4) gives

$$0 \leqslant p(gu - fu)$$
$$\leqslant p(rgu + (1 - r)fu - fu)$$
$$\leqslant rp(gu - fu)$$

which is absurd if $p(gu - fu) \neq 0$. Hence (7) holds.

**Remark :**

If $E$ is a complex vector space and $X$ is a convex set the inwardness condition (5) may be weakened as follows : $f$ is an inward map if ($5^1$) $x \in X$ implies that $\{y \in E : y = rx + (1\text{-}r)fx, |r| < 1\} \cap X \neq \phi$. (In case $E$ is a real vector space, condition ($5^1$) is easily shown to be equivalent to condition (5)).

The $g$-inwardness condition (6) may be weakened in the same way. The map $f$ is g-inward if

($6^1$) $x \in X \rightarrow \{y \in E : y = rgx + (1\text{-}r)fx, |r| < 1\} \cap X \neq \phi$.

If $f$ satisfies ($6^1$), Theorem 3 (Theorem 4) still holds.

In order to prove that our results contain the more general theorems of coincidence and fixed points known till now in normed spaces, locally eonvex spaces are topological vector spaces, we will use the following

**Lemma :**

Let $E$ be a topological vector space with sufficiently many linear functionals. If $H \subset E$ is a weakly compact set and $0 \notin H$, then there exists a seminorms $p$, continuous in the weak topology of $E$, such that $p(x) \neq 0$ for all $x \in H$.

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**Proof :**

For every $x \in H$ there exists a continuous linear functional $x'_x \in E'$ and a $w$-open neighborhood $N(x)$ of $x$ such that $| x'_x (y) | > 0$ for alt $y \in N(x)$ .

Let $\{ N(x_i) \}$, $i = 1, 2, \ldots, n$ be a finite subcovering of the covering $\{ N(x) \}_{x \in H}$ of $H$, Then the seminorm $p(x) = \sum_1^n | x'_{x_i} (x) |$ is positive for every $x \in H$. Moreover, for every $x \in E$ and $\epsilon > 0$ if $y \in N$, where $N = \{ z \in E : | x'_{x_i} (z-x) | < \frac{\epsilon}{n}$ , $i=1, 2, \ldots, n \}$ , we have $p(y) < \varepsilon$ . Tnen $p$ is continuous in the weak topology of $E$ .

Since a strongly compact subset of $E$ is also weakly compact and since the seminorm $p$ of tke Lemma is obviously continuous in the strong topology of $E$, it is easy to see that Theorem 3 contains as a particular case . Theorem 5 of Hadzic [4] and Theorem 4 conrains the Theorem of Kaczynski [5] . ( Recall remark after Theorem 3 and assume $g(x) = x$ ) .

We may also obtain new results . As an examle, Theoren 4 and Lemma give the following :

**Theorem (5) :**

Let $(E, \tau)$ be a topological vector space with sufficently many linear functionals, $K \subset E$, a non-empty, convex, $\tau$-compact set, $T$ the weak topology of $E$ and $f : (X, \tau) \to (E, T)$ be a continuous inward map. Then $f$ has a fixed point in $K$ .

**Theorem 6.**

Let $(E, \tau)$ be a topologieal vector space with sufficiently many linear functionals, $T$ the weak topology of $E$, $K \subset E$ a non-empty,

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convex, $T$-compact set, $f : (K, T) \to (E, \tau)$ a continuous inward map. Then $f$ has a fixed point in $K$.

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# A FIXED POINT THEOREM FOR CONTRACTION TYPE MAPPINGS IN CONVEX SPACES

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*( Received : June 15, 1988 )*

## ABSTRACT

The result herein provides a common fixed point theorem for a pair of contraction type mappings with respect to a bounded closed and convex subset of a locally convex space. Theorem 1 below extends an earlier result of Dubinsky and the authors ,

Let $E$ be a separated locally convex topological vector space and $M$ a nonempty convex subset of $E$ containing the origin 0 . Let $B$ be a closed, bounded and convex subset of $S$ and $\phi : [0, \infty) \to [0, \infty)$ a mapping with $\phi(t) \leqslant t$ for $t$ close to zero . Let $C(E)$ denote the nonempty closed subsets of $E$.

**Definition .**

Let $x_0 \in E$ and for $i \in \{1, 2\}$, let $f_i : x_0 + M \to C(E)$ be a pair of mappings . The pair $(f_1, f_2)$ is a $\phi$-contraction iff for each $x, y$ in $x_0 + M$ with $y - x \in \lambda B$ for some $\lambda \geqslant 0$ aud $u \in f_i(x)$ implies that $(f_J(y) - u) \cap \phi(\lambda) B \neq \phi, i \neq J$ . Note that the definition implies

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that for $u \in f_j(x)$ there exists a $v \in f_j(y)$, $j \neq i$ with $v-u \in \phi(\lambda) B$.

**Theorem 1.**

Let $E$ be sequentially complete and for $i=1, 2$, let $f_i : x_0 + M \to C(E)$ be a pair mappings. If the pair $(f_1, f_2)$ is a $\phi$ - contraction and for some k, $0 \leqslant k < 1$

(1) $(f_1(x_0) - x_0) \cap kB \neq \phi$

(2) $\sum_{p=0}^{\infty} \phi^p(k) = a \leqslant 1$ where $\phi^0(k) = k$,

then $f_1, f_2$ have a common fixed point, that is there exists a $u \in x_0 + M$ with $u \in f_1(u) \cap f_2(u)$.

**Proof .**

First, we show by induction the existence of a sequence $\{x_n\}$ satisfying

(a) $x_n \in x_0 + M$ for each $n$,

(b) $x_n \in \begin{cases} f_1(x_{n-1} \text{ if } n \text{ is an odd integer} \\ f_n(x_{n-1}) \text{ if } n \text{ is even} \end{cases}$

(c) $x_n - x_{n-1} \in \phi^{n-1}(k) B$ for each $n$.

Clearly by (1) there exists a $x_1 \in f_1(x_0)$ with $x_1 - x_0 \in kB \subseteq M$. Assume that for some fixed $n$, $x_i$, $i=1, 2, \ldots n$ have been obtained satisfying (a), (b) and (c) above. Without loss of generality we may assume that n is an even integer. This implies that

$$x_n \in (x_0 + M) \cap f_2(x_{n-1}) \text{ and}$$

$$x_n - x_{n-1} \in \phi^{n-1}(k) B \subseteq M.$$

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since $x_n \in f_2(x_{n-1})$, it follows by definition that there exists a $x_{n+1} \in f_1(x_n)$ with $x_{n+1} - x_n \in \phi^n(k) B$. To complete the induction, we need to show that $x_{n+1} \in x_0 + M$. To prove this, note that since $B$ is convex

$$(3) \quad x_{n+1} - x_0 = \sum_{p=0}^{n} (x_{p+1} - x_p) \in \sum_{p=0}^{n} \phi^p(k) B$$

$$\subseteq \left( \sum_{p=0}^{n} \phi^p(k) \right) B \subseteq aB \subseteq M.$$

Thus, $x_{n+1} \in x_0 + M$. This completes the proof of the existence of the stated sequence $\{x_n\}$.

Now, we show that the sequence $\{x_n\}$ thus constructed is infact a Cauchy sequence. Let $V$ be an arbitrary convex neighborhood of the origin. Since, $B$ is a bounded set, choose an $\varepsilon_0 > 0$ sueh that $\varepsilon B \subseteq V$ and olso satisfies $\phi(\varepsilon) \leqslant \varepsilon$ for $0 \leqslant \varepsilon \leqslant \varepsilon_0$. Let $N$ be any integer for which $\sum_{p=N}^{\infty} \phi^p(k) \leqslant \varepsilon_0$.

Let $\lambda_n = \sum_{p=n}^{\infty} \phi^p(k)$. For $m \geqslant n \geqslant N$,

$$(4) \quad x_m - x_n = \sum_{p=n}^{m-1} (x_{p+1} - x_p) \in \left( \sum_{p=n}^{m-1} \phi^p(k) \right) B \subseteq \lambda_n B \subseteq V.$$

This proves that the sequence $\{x_n\}$ is Cauchy in $E$. Consequently, there is a $u \in E$ such that $x_n \to u$. Further, since $B$ is closed, it follows from (3) that $u - x_0 \in aB$, that is $u \in x_0 + M$ and (4) implies that for all $n \geqslant N$, $u - x_n \in \lambda_n B$. In particular for $n \geqslant N$

*a*) $u - x_{2n} \in \lambda_{2n} B$ and *b*) $u - x_{2n+1} \lambda_{2n+1} B$.

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We show that $(a)$ implies that $u \in f_2(u)$. Since $x_{2n+1} \in f_1(x_{2n})$ it follows by $(a)$ that for each $n \geqslant N$, there is a $u_n \in f_2(u)$ satisfying

$$u_n - x_{2n+1} \in \phi(\lambda_{2n}) B \subseteq \lambda_{2n} B .$$

Since $\lambda_{2n} \to 0$ as $n \to \infty$, it follows as in (4) that $u_n - x_{2n+1} \to 0$, that is $u_n \to u$, However, $u_n \in f_2(u)$ and $f_2(u)$ is closed. Consequently, $u \in f_2(u)$ .

In a similar way, it follows from (b) that $u \in f_1(u)$ .
Thus $u \in f_1(u) \cap f_2(u)$.

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# UNIFORM SUMMABILITY-Λ OF ORTHONORMAL EXPANSIONS OF SUMMABLE FUNCTIONS

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## ABSTRACT

In this paper, we establish necessary and sufficient conditions for the uniform sumn ability-Λ, of the orthonormal expansions of summable functions .

Let $f(x)$ be an integrable function and $\{\phi_\nu(x)\}$, $\nu=0, 1, 2, \ldots$ be an orthonrmal system in $[0, 1]$.

Let

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu \phi_\nu(x),$$

where

$$c_\nu = \int_0^1 f(t)\, \phi_\nu(t)\, dt \quad (\nu = 0, 1, 2, \ldots),$$

be an orthonormal expansion of $f(x)$ in the system $\{\phi_\nu(x)\}$.

Let $\{\lambda_\nu\}$ be an infinite sequence of real numbers,

If the series

$$\sum_{\nu=0}^{\infty} \lambda_\nu c_\nu \phi_\nu(x)$$

converges uniformly in $x$, then $\{\lambda_\nu\}$ is called a sequence of uniform convergence factors of orthonormol expansions of $f(x)$.

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If

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu \phi_\nu(x)$$

then

$$S_n(f;x) = \sum_{\nu=0}^{n} \lambda_\nu c_\nu \phi_\nu(x) = \int_0^1 f(t)\left( \sum_{\nu=0}^{n} \lambda_\nu \phi_\nu(x)\ \phi_\nu(t) \right) dt$$

$$= \int_0^1 f(t)\, K_n(x, t)\ dt$$

where

$$K_n(x, t) = \sum_{\nu=0}^{n} \lambda_\nu \phi_\nu(x)\, \phi_\nu(t),$$

In this paper we consider an orthonormal system $\{\phi_\nu(x)\}$ closed in $c\,[0, 1]$, the class of continuous functions on $[0, 1]$, and $|\phi_\nu(x)| < A_\nu$, independent of $x$.

In [3] we proved the following theorem ,

**Theorem 1.** A necessary and sufficient condition that $\{\lambda_\nu\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L$ (summable functions), is that there exists an $M$ such that

$$\underset{0 \leqslant t \leqslant 1}{ess\ sup} |K_n(x, t)| < M$$

for all $x$ and $n$.

Here ess sup lends a symmetry to the result and actually it is not needed, since $|\phi_\nu(x)| < A_\nu$ for all $\nu$ .

Now let

$$\| \lambda_{\nu\mu} \| \quad (\nu, \mu = 0, 1, 2, \ldots)$$

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be an infinite matrix of real elements .

If

$$\Lambda_{\mu,x}(f) = \sum_{\nu=0}^{\infty} \lambda_{\nu\mu}\, c_\nu\, \phi_\nu(x)$$

is uniformly convergent in $x$, for each $\mu = 0, 1, 2, \ldots$ and

$$\lim_{\mu\to\infty} \Lambda_{\mu,x}(f) = \Lambda_x(f)$$

exists uniformly in $x$, then orthonormal expansions of $f(x)$ are said to be uniformly summable-$\Lambda$.

The $\Lambda$-summation method is not necessarily regular, *i. e.*, every convergent series is not summed by it to its sum in the sense of convergence.

We note if

$$f(x) \sim \sum_{\nu=0}^{\infty} c_\nu\, \phi_\nu(x)$$

then

$$\Lambda_{n,\mu,x}(f) = \sum_{\nu=0}^{n} \lambda_\nu\, c_\nu\, \phi_\nu(x) = \int_0^1 f(t) \left( \sum_{\nu=0}^{n} \lambda_{\nu\mu}\, \phi_\nu(x)\, \phi_\nu(t) \right) dt$$

$$= \int_0^1 f(t)\, K_{n,\mu}(x, t)\, dt$$

where

$$K_{n,\mu}(x, t) = \sum_{\nu=0}^{n} \lambda_{\nu\mu}\, \phi_\nu(x)\, \phi_\nu(t) .$$

We may write

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$$\Lambda_{\mu,x}(t) = \sum_{\nu=0}^{\infty} \lambda_{\nu\mu}\, c_\nu\, \phi_\nu(x) = \lim_{n\to\infty} \Lambda_{n,\mu,x}(f)$$

$$= \lim_{n\to\infty} \int_0^1 f(t)\, K_{n,\mu}(x, t)\, dt\,.$$

Formally, we write

$$\Lambda_{\mu,x}(f) = \int_0^1 f(t) \left( \sum_{\nu=0}^{\infty} \lambda_{\nu\mu}\, \phi_\nu(x)\, \phi_\nu(t) \right) dt = \int_0^1 f(t)\, K_\mu(x,t)\, dt.$$

We do not mean

$$K_\mu(x, t) = \sum_{\nu=0}^{\infty} \lambda_{\nu\mu}\, \phi_\nu(x)\, \psi_\nu(t)$$

in the sense of ordinary convergence, but rather a formal relationship which can be made precise by a suitable definition for the sum of the series .

We first need a lemma .

**Lemma 1.** If $\{ g_n(x) \}$ is a sequence in $B$, the space of measurable bounded functions, such that for every $f(x) \in L$

$$\lim_{n\to\infty} \int_0^1 f(x)\, g_n(x)\, dx$$

exists, and $| g_n(x) | \leqslant C$ almost everywhere, then there exists a $g(x) \in B$, $| g(x) | \leqslant C$ almost everywhere such that

$$\lim_{n\to\infty} \int_0^1 f(x)\, g_n(x)\, dx = \int_0^1 f(x)\, g(x)\, dx$$

for every $f(x) \in L$.

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**Proof :** If $\{ g_n (x) \} \in B$, then $\{g_n (x) \} \in L$ and by hypothesis

$$\lim_{n\to\infty} \int_0^1 f(x)\, g_n (x)\, dx$$

exists for every $f(x) \in B$:

Therefore, $\{ g_n (x) \}$ converges weakly in $L$ and since $L$ is weakly complete ( [2], p. 240 ) there exists a $g(x) \in L$ such that $g_n (x)$ converges weakly to $g(x)$ in $L$.

Now suppose

$$| g(x) | > C$$

on a set $H$ of positive measure, *i. e.*, $m(H) > 0$ .

Let $X(x)$ be the characteristic function of $H$ ,

Then

$$\lim_{n\to\infty} \int_0^1 X(x) ( g_n (x) - g(x) ) = \lim_{n\to\infty} \int_H ( g_n (x) - g(x) )\, dx > 0,$$

which contradicts the weak convergence of $g_n (x)$ to $g(x)$ in $L$ , Hence

$$| g(x) | \leqslant C .$$

Now let $f(x) \in L$, we must show that

$$\lim_{n\to\infty} \int_0^1 f(x)\, g_n (x)\, dx = \int_0^1 f(x)\, g(x)\, dx .$$

For every $\varepsilon > 0$, we decompose $f(x)$ such that

$$f(x) = f_1 (x) + f_2 (x)$$

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where

$$| f_1 (x) | \leq M$$

and

$$\int_0^1 | f_2 (x) | dx < \varepsilon .$$

then

$$\left| \int_0^1 f(x) (g_n (x) - g (x)) dx \right| = \int_0^1 ( f_1 (x) + f_2 (x) ) ( g_n (x) - g(x) ) dx \Bigg|$$

$$\leq \left| \int_0^1 f_1 (x) ( g_n (x) - g (x) ) dx \right| + \left| \int_0^1 f_2 (x) ( g_n (x) - g(x) ) dx \right|$$

$$< \varepsilon + 2 C \varepsilon$$

for sufficiently large $n$ .

Hence

$$\lim_{n \to \infty} \int_0^1 f(x) g_n (x) dx = \int_0^1 f (x) g (x) dx$$

for every $f (x) \in L$ .

The proof of Lamma 1 is complete .

Now, from Lemma 1, there exists a $K_\mu ( x, t ) \in B$ such that

$$\lim_{n \to \infty} \int_0^1 f (t) K_{n,\mu} ( x, t ) dt = \int_0^1 f(t) K_\mu ( x, t ) dt$$

for every $f ( t) \in L$ .

We note that $\{ K_{n,\mu} (x,t) \}$ meets the requirements of Lemma 1 by

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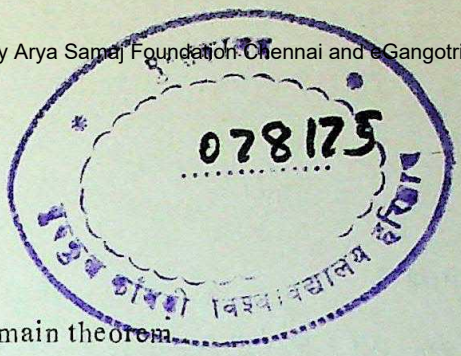



**Thorem 1.**

Now we get our main theorem.

**Theorem 2.** The necessary and sufficient conditions for the uniform summability-$\Lambda$ of orthonormal expansions of $f(x) \in L$, are

(1) $$\lim_{\mu \to \infty} \lambda_{\nu\mu}$$

exists for each $\nu = 0, 1, 2, \ldots$

(2) $$ess\ sup_{0 \leqslant t \leqslant 1} | K_{n,\mu}(x, t) | < M_\mu \quad (\mu = 0, 1, 2, \ldots)$$

independent of $n$ and $x$.

(3) $$ess\ sup_{0 \leqslant t \leqslant 1} | K_\mu(x, t) | < M$$

independent of $\mu$ and $x$.

To prove the theorem we use a lemma of Banach-Steinhaus type ( [3] ).

Lemma 2. Let $\{ A_{n,x}(f) \}$ be a sequence of linear functionals defined on the Banach space $X$, and $Y$ be an everywhere dense subset of $X$.

In order that $\lim_{n \to \infty} A_{n,x}(f)$ exists uniformly in $x$ ($x$ belonging to a set $E$) for every $f \in X$, it is a necessary and sufficient that

(A) there exists an $N$ and an $M$ such that

$$\| A_{n,x} - A_{m,x} \| < M$$

for all $x$, and $n, m > N$.

(B $\lim_{n \to \infty} A_{n,x}(g)$ exists uniformly in $x$, for every $g \in Y$.

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**Proof of Theorem 2.**

Clearly $\Lambda_{\mu,x}(f) = \lim_{n\to\infty} \Lambda_{n,\mu,x}(f)$ is a continuous linea functional on $L$.

Now for each $\mu = 0, 1, 2, \ldots$, the sequence $\{\lambda_{\nu,\mu}\}$ be a sequence of uniform convergence factors of orthonormal expansions of $f(x) \in L$ the condition (2) is necessary and sufficient by Theorem 1.

We show that condition (B) of Lemma 2 is satisfied in the theorem. Since $\{\phi_\nu(x)\}$ is closed in $C$, the finite linear combinations

$$g(x) = \sum_{\rho=0}^{k} \gamma_\rho \phi_\rho(x),$$

are dense in $C$ and hence dense in $L$.

From the orthonormality of $\{\phi_\nu(x)\}$ we get

$$\Lambda_{\mu,x}(g) = \lim_{n\to\infty} \Lambda_{n,\mu,x}(g)$$

$$= \lim_{n\to\infty} \int_0^1 \Big(\sum_{\rho=0}^{k} \lambda_\rho \phi_\rho(t)\Big)\Big(\sum_{\nu=0}^{n} \lambda_{\nu\mu} \phi_\nu(x) \phi_\nu(t)\, dt$$

$$= \sum_{\rho=0}^{k} \gamma_\rho \lambda_{\rho\mu} \phi_\rho(x).$$

Hence if

$$\lim_{\mu\to\infty} \Lambda_{\mu,x}(g)$$

exists uniformly in $x$, then

$$\lim_{\mu\to\infty} \lambda_{\nu\mu}$$

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exists for each $\nu = 0, 1, 2, \ldots$

So condition (B) of Lemma 2 implies condition (1) of the theorem.

Now consider

$$\left| \Lambda_{\mu', x}(g) - \Lambda_{\mu'', x}(g) \right| = \left| \lim_{n \to \infty} \int_0^1 g(t) \left( K_{n, \mu'}(x, t) - K_{n, \mu''}(x, t) \right) dt \right| = \left| \sum_{\rho=0}^{k} \gamma_\rho \left( \lambda_{\rho\mu'} - \lambda_{\rho\mu''} \right) \phi_\rho(x) \right|$$

$$\leqslant \sum_{\rho=0}^{k} \left| \gamma_\rho \right| \left| \lambda_{\rho\mu'} - \lambda_{\rho\mu'''} \right| \left| \phi_\rho(x) \right|$$

$$\leqslant \sum_{\rho=0}^{k} \left| \gamma_\rho \right| \left| \lambda_{\rho\mu'} - \lambda_{\rho\nu''} \right| A_\rho .$$

Hence, if

$$\lim_{\mu \to \infty} \lambda_{\nu\mu}$$

exists for each $\nu = 0, 1, 2, \ldots$ then

$$\lim_{\mu \to \infty} \Lambda_{\mu, x}(g)$$

exists uniformly in $x$, for ezery $g$ belonging to dense subset of $L$.

Thus condition (B) of Lemma 2 is equivalent to condition (1) of the theorem.

We note

$$\left| \left| \Lambda_{\mu, x} \right| \right| = \operatorname{ess\,sup}_{0 \leqslant t \leqslant 1} \left| K_\mu(x, t) \right| .$$

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We show that condition (A) of Lemma 2 is equivalent to uniform boundedness of norm, *i e*,

$$\| \Lambda_{\mu,x} \| \leqslant M$$

for all $\mu$ and $x$, hence equivalent to condition (3) of the theorem. Assume

$$\| \Lambda_{\mu,\, x} \| < M$$

for all $x$, and $\mu > N$.

We get $\| \Lambda_{\mu}', x - \Lambda_{\mu}'', x \| \leqslant \| \Lambda_{\mu}', x \| + \| \Lambda_{\mu}'', x \| < 2M$

for all $x$, and $\mu', \mu'' > N$.

So condition (1) of the theorem implies condition (A) of Lemma 2. Now since $| \phi_{\nu} (x) | < A_{\nu}$, independent of $x$,

$$| K_{\mu} ( x, t ) | < C_{\mu}$$

independent of $x$ and $t$.

Consider

$$\left| \Lambda_{\mu,\, x} ( f ) \right| = \left| \int_0^1 f(t)\, K_{\mu} ( x, t )\, dt \right| \leqslant \| f \|\ ess \mid sup\ K_{\mu} ( x, t ) \mid$$

$$\leqslant \| f \| C_{\mu}.$$

Therefore,

$$\| \Lambda_{\mu,\, x} \| < C_{\mu}.$$

Now condition (A) of Lemma 2 gives

$$\| \Lambda_{\mu}', x - \Lambda_{\mu}'', x \| < M$$

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for all $x$, and $\mu', \mu'' \geq N$.

So

$$\| \Lambda_{\mu'}, x \| \leqslant \| \Lambda_{\mu''}, x \| + M$$

$$\leqslant C_{\mu''} + M = M'$$

for all $x$, and $\mu' > N'$.

Hence, condition (A) of Lemma 2 is equivalent to condition (3) of our theorem.

Thus we have shown that conditions (1) and (3) of our theorem are necessary and sufficient for the uniform convergenee of $\Lambda_{\mu,x}(f)$,

The proof of Theorem 2 is complete.

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# COLLINEATION INDUCED INCIDENCE RELATIONS*

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## 1. INTRODUCTION

The triangle has associated with it a number of well known remarkable points. While each of these possess individual properties important in the geometry of the triangle, certain subsets share the further property of lying on the same line . Some of these, such as the Euler line which contains the centroid, circumcentre, orthocentre, and nine-point centre of the given triangle, have been known for some time while others, described later, are more recent. In this paper, we apply simple collineations to some known incidence relations and obtain others believed to be new . Interestingly, all of the latter except one, involve a seemingly little known point, the "middlespoint" that appears in a paper by C. H. von Nagel published in 1836.

As a secondary objective, we attempt to demonstrate how classical Euclidean geometry of the plane may be enriched and made more interesting by the use of other techniques, in this case, those of projective geometry, In return, material that is essentially Euclidean, but which has a natural link with real projective geometry, can help make more concrete, and hence more appealing, the abstract area of general projective geometry. Since the latter, with its attention to

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rigor, generally takes precedence over the Euclidean in present-day curricula, we shall emphasize the second interaction.

## 2. Background.

In the Euclidean plane $E^2$, any two points determine a unique line This also holds in the projective plane $P^2$ where we have, additionally, the dual statement every pair of lines determine a unique point and parallelism is not an issue. While this more general setting can not be handled by the usual Cartesian coordinate system, a related system works quite nicely.

Consider the equations

$$x = \frac{x_1}{x_3}, \quad y = \frac{x_2}{x_3}$$

where $x_i$, $i = 1, 2, 3$, are real and $x_3 \neq 0$, then each ordered triple $(x_1, x_2, x_3)$ has the ordered pair $(x, y)$ as a unique mate in $E^2$ when only the ratios $x_1 : x_2 : x_3 = x : y : 1$ are considered. The triple $(x_1, x_2, x_3)$ thus represents the homogeneous coordinates of a point $p$ in the projective plane. Since all non-zero scalar multiples of a given triple represent the same point, we have the added convenience of working with integer coordinates. For example, the mate of the point $\left(\frac{1}{2}, \frac{-2}{7}\right)$ in $E^2$ has coordinates $\left(\frac{1}{2}, \frac{-2}{7}, 1\right) = (7, -4, 14)$ in $p^2$. The points $(x_1, x_2, 0)$ have no mates in $E^2$ and thus constitute the line at infinity, $l_\infty : x_3 = 0$, in $p^2$.

There is, in general, on projective counterpart for the Cartesian distance formula thus projective geometry with such non-metric notions as straightness and tangency One can include parallelism in this non-metric framework by singling out the line at infinity and

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defining a pencil of parallel lines as one with its centre on this line. This geometry, which is sort of in between Euclidean and projective, is called affine geometry.

A fundamental property of the projective plane is that it has a collineation structure, *i. e.* there exists a map called a collineation, that preserves straight lines ( point ranges ) and line pencils. This map, defined by the rule

$$\rho x'_i = \sum_j a_{ij} x_j \; ; \; i, \; j = 1, 2, 3, \qquad (1)$$

where the determinant $| a_{ij} | \neq 0$ and $\rho$ is a non-zero constant, maps collinear points to image sets which are also collinear . when dealing with the geometry of the triangle, one needs to use the measures of the sides and angles of the given triangle. Two systems of specialized homogeneous coordinates that do this, while at the same time retain all the advantages of the general homogeneous type, are the trilinear (or normal) system and the barycentric (or areal) system.

We now provide a brief description of each of these systems, see [5] or [7]. In the trilinear case, the coordinates of a point $P\,(x_1, x_2, x_3)$ in the plane are proportional to the signed distances $d_1, d_2, d_3$ of $P$ from the sides of a given triangle of reference $A_1A_2A_3$ . The vertices of this triangle have coordinates representations $A_1$ ( 1, 0, 0 ), $A_2$ ( 0, 1, 0 ), $A_3$ ( 0, 0, 1 ) and the unit point of the system is the incentre $I=(r,r,r) = (1,1,1)$, where $r$ is the inradius of $A_1A_2A_3$. Tee distance $d_i$, $i=1,2,3$ is positive if $p$ is in the half-plane determined by $I$ and $A_jA_k$, $i \notin \{ j, k \}$, and negative otherwise. The excentre $I_1$ opposite vertex $A_1$, for example, has coordinates ( - 1, 1, 1 ) or its projective equivalent ( 1, - 1, - 1 ). In the barycentric case, the coordinates of $P$ ( $y_1, y_2, y_3$ ) are proportional to the signed areas of the triangle $PA_2A_3, PA_3A_1, PA_1A_2$ with the signs being determined as in the trilinear case with

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the unit point of the system being the centroid $S$. The line at infintiy is the unit line $x_1 + x_2 + x_3 = 0$, while for the trilinear case it is $a_1x_1 + a_2x_2 + a_3x_3 = 0$, where $a_i$, $i = 1, 2, 3$ are the measures of the sides of the reference triangle. We shall use this system for the remainder of the paper.

Incidence relations are readily determined by such a system. Consider the points $P_1 ( x_1, x_2, x_3 )$, $P_2 ( y_1, y_2, y_3 )$, $P_3 ( z_1, z_2, z_3 )$ respectively ; then $P_1, P_2, P_3$ are collinear if and only if the determinantal equation

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \qquad (2)$$

is true.

For concurrency of lines, there exists a dual procedure. If, for example, we represent the line $l_1 : u_1x_1 + u_2x_2 + u_3x_3 = 0$ by its "line" or "pluckerian" coordinates $l_1 ( u_1, u_2, u_3 )$. see [6], then $l_1$, $l_2 ( v_1, v_2, v_3 )$, and $l_3 ( w_1, w_2, w_3 )$ are concurrent if and only if

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0 . \qquad (3)$$

In Table 1, we list several of the special points in the plane associated with the triangle of reference $A_1A_2A_3$ along with their coordinate representations.

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**Table 1**

| Point | Symbol | Trilinear Coordinatea |
|---|---|---|
| Incentre | $I$ | $(1, 1, 1)$ |
| Excentre | $I_{A_1}$ | $(-1, 1, 1)$ |
| Centroid | $S$ | $(\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3})$ |
| Circumcentre | $O$ | $(\cos \alpha_1, \cos \alpha_2, \cos \alpha_3)$ |
| Orthocentre | $H$ | $(\frac{1}{\cos \alpha_1}, \frac{1}{\cos \alpha_2}, \frac{1}{\cos \alpha_3})$ |
| Lemoine point | $K$ | $(a_1, a_2, a_3)$ |
| Nagel point | $G$ | $(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_2})$, $s = \frac{a+b+c}{2}$ |
| Gergonne point | $N$ | $(\frac{1}{a_1(s-a_1)}, \frac{1}{a_2(s-a_2)}, \frac{1}{a_3(s-a_3)})$ |
| Middlespoint | $M$ | $(s-a_1, s-a_2, s-a_3)$ |

The derivation of these coordinates is especially easy when the segments, which the concurrent lines through the given point and the vertices of $A_1A_2A_3$ cut the opposite sides, are known. Consider the point $P$ and the pencil of lines $A_1PD_1$, $A_2PD_2$, $A_3PD_3$, where $D_1$ is on $A_2A_3$, etc., see figure 1, then for the point

$D_1$, $d_2 : d_3 = \frac{\overline{D_1A_3}\sin\alpha_3}{\overline{A_2D_1}\sin\alpha_2}$, where $\alpha_1$, $\alpha_2$, $\alpha_3$ denote the measures of the angles at $A_1$, $A_2$, $A_3$ respectively, see Table 1. Using the well known expressions $a_2 = 2R\sin\alpha_2$ and $a_3 = 2R\sin\alpha_3$, where $R$ is the circumradius of $A_1A_2A_3$, which relate the measures of the sides

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and angles of the given triangle, we obtain

$$D_1 = ( 0, a_3 \overline{D_1 A_3}, a_2 \overline{A_2 D_1} ) \qquad (4)$$

as the trilinear coordinates of $D_1$. Since the coordinates for $D_2$ and $D_3$ may be derived in a similar manner, the coordinates of $P$ are immediate.

To illustrate the idea, we consider three points from table 1 that might not be widely known and indicate briefly how to derive their corresponding coordinates All three are readily derived as above since the positions of the $D_i$, $i=1, 2, 3$, in each case are known. We first consider the Lemoine or symmedium point $K$ where the $D_i$'s are such that

$$\frac{A_j D_i}{D_i A_k} = \frac{a_k^2}{a_j^2}, \quad i, j, k = 1. 2, 3 .$$

In this case, $D_1 = ( 0, a_2, a_3 )$ with similar representations for $D_2$ and $D_3$, hence $K = ( a_1, a_2, a_3 )$.

Also fitting this picture are the Gergonne and Nagel points $G$ and N shown in figures 2 and 3 respectively.

The coordinates of each of these points now follow readily since for $G$, $\overline{A_2 D_1} = s - a_2$ and for $N$, $\overline{A_2 D_1} = s - a_3$, etc.

We now come to the last point in table 1 and one which plays a key role in this paper. see figure 4.

*Let* $S_1$, $S_2$, $S_3$ denote the midpoints of $\overline{A_2 A_3}$, $\overline{A_3 A_1}$, $\overline{A_1 A_2}$ respectively, then the lines $I_1 S_1$, $I_2 S_2$, $I_3 S_3$ are concurrent at the middlespoint ( mittenpunkt ), $N$ of the given triangle, see Nagel [4].

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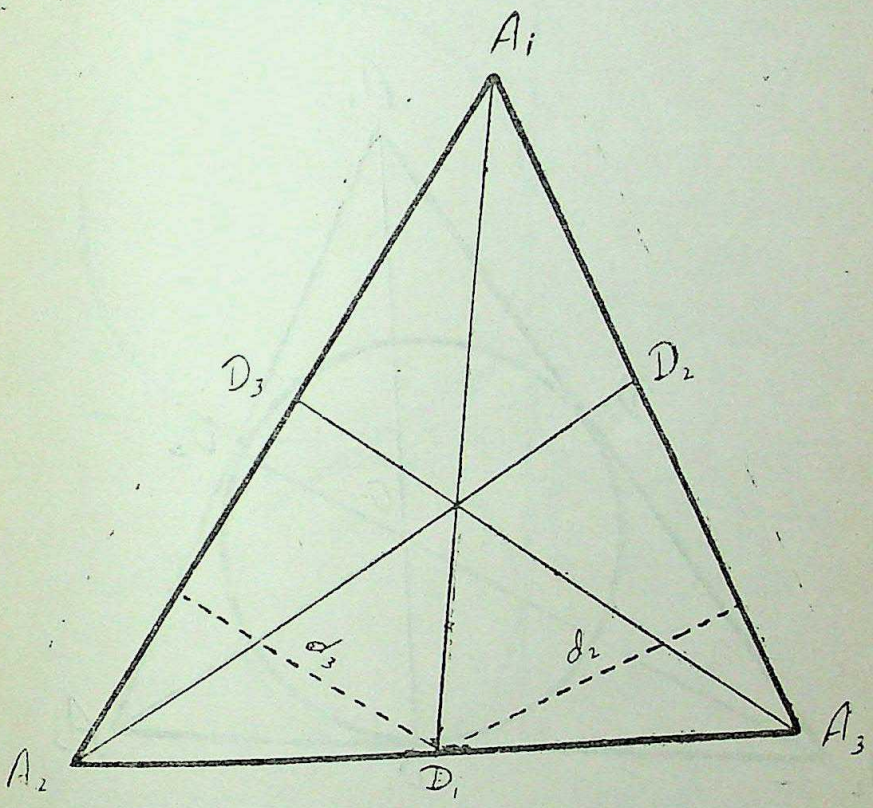


**FIGURE 1**

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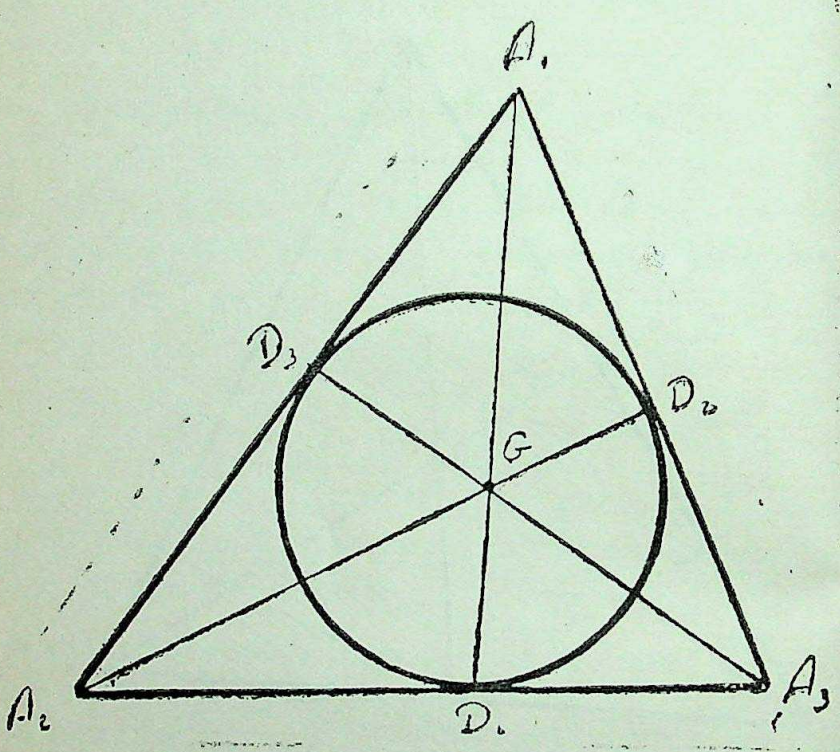


FIGURE 2

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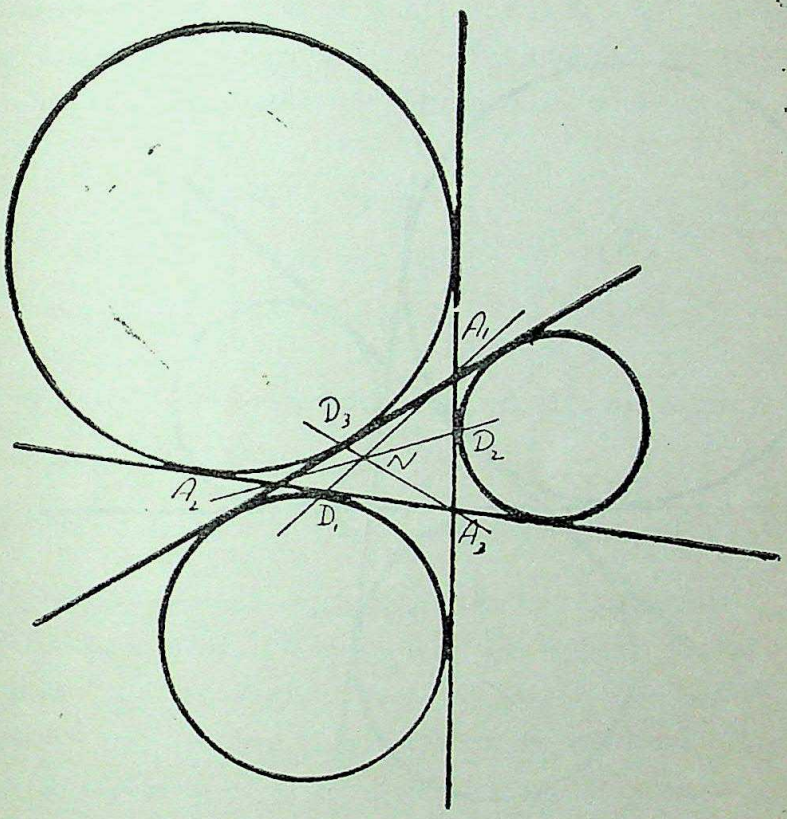


FIGURE 3

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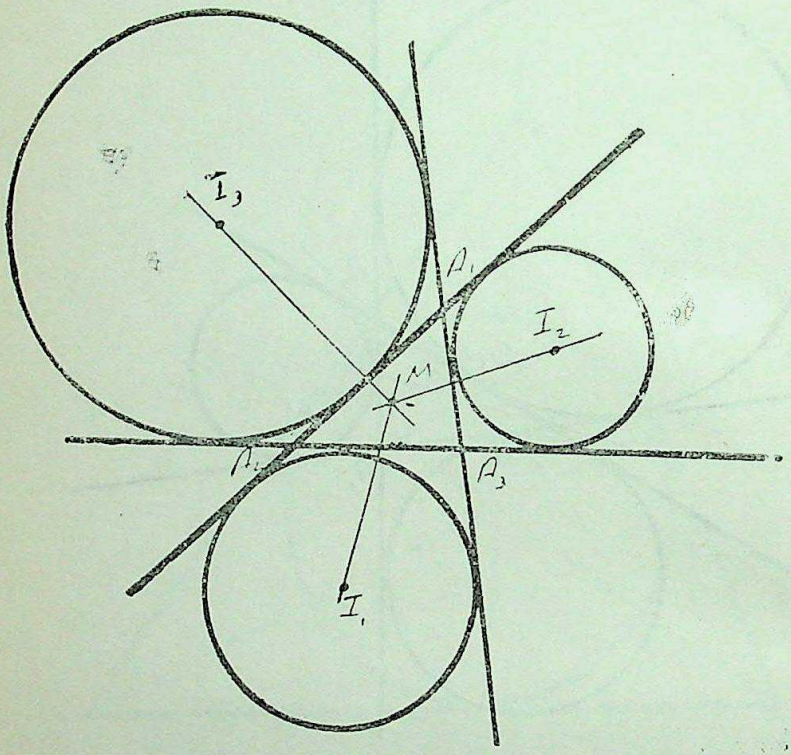


FIGURE 4

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Again, the derivation of the coordinates is straightforward. The lines $I_1S_1$, $I_2S_2$, $I_3S_3$ are easily seen to have pluckerian coordinates $(a_2-a_3, a_2,-a_3)$, $(-a_1, a_3-a_1, a_3)$, $(a_1, -a_2, a_1-a_2)$, consequently, the concurrency property follows immediately by (3). Eor the coordinates of $M$, consider the determinantal equation

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ a_2-a_3 & a_2 & -a_3 \\ -a_1 & a_3-a_1 & a_3 \end{vmatrix} = 0, \qquad (5)$$

where $u_1$, $u_2$, $u_3$ are the coordinates of a general line in the plane.

Expanding, we obtain the pencil.

$$(a_2 + a_3 - a_1)\,u_1 + (a_2 + a_1 - a_2)\,u_2 + (a_1 + a_2 - a_3)\,u_3 = 0 \qquad (6)$$

with centre $M = (s - a_1, s - a_2, s - a_3)$. This interesting point, which perhaps derives it name from the fact that its construction involves "middles" *i. e. centres* of circles and the *midpoints* of line segments, seems to have been discovered by Nagel and, surprisingly, does not appear to have found its way into the popular literature dealing with the geometry of the triangle While some points may be "born to blush unseen" either because of a complicated construction or a cumbersome coordinate representation, this is certainly not the case with the middlespoint. The only other paper known to us that deals with this point is a paper by Baptist [1], soon to appear, which contains, in particular, proposition 1 below.

Other special points may be obtained from the known ones by applying two quadratic transformations. As will be seen, these turn out to be involutions that are well-defined for all poiuts in the

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plane except those incident with the triangle of reference, The first of these is the "isogonal" transformation defined as follows.

*If the lines $A_i D_i'$, $i=1, 2, 3$, through $P$ are each reflected in the angle bisector through the same vertex, the reflected lines $A_i D_i'$ are concurrent at $P^g$, ihe isogonal conjugate of $P$, see [3].*

It is an immediate consequence of the definition thati f $P$ has trilinear coordinates $( x_1, x_2, x_3 )$, then $P^g$ has coordinates

$$\left( \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3} \right) = ( x_2 x_3, x_3 x_1, x_1 x_2 ) .$$ Two such sets, immediately seen from Table 1, are $\{O, H\}$ and $\{ S, K \}$.

The second transformation is called "isotomic" and is defined as in the following manner.

*If the points $D_i'$ are the reflections of $D_i$ in the midpoints $S_i$ of the sides of the triangles $A_1A_2A_3$, the lines $A_iD_i'$ concur at $P^t$, the isotomic conjugate of $P$, see [3].*

Again, it is straightforward to ascertain that the trilinear coordinates of $P^t$ are $\left( \frac{1}{a_1^2 x_1}, \frac{1}{a_2^2 x_2}, \frac{1}{a_3^2 x_3} \right)$. It is readily seen from table 1 that $\{ G, N \}$ is a pair of isotomic conjugates.

**3. The results.** We are now in a position to derive results believed to be new. As indicated in the title, these, will be obtained by applying certain collineations to known incidence relations stated in the form of propositions. Each of the collineations is defined by a matrix $B = ( b_{ij} )$; $i, j = 1, 2, 3$, which is diagonal with $b_{11} . b_{22}$, $b_{33} \neq 0$. In general, this clsss of transformation forms a group

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which fixes the triangle of reference and is sharply transitive on the points of the plane not incident with its sides . Thus, for convenience we represent each collineation by the general diagonal element $b_{ii}$, $i = 1, 2, 3$ of its defining matrix .

**PROPOSITION 1.** The incentre, Nagel, point, and the centroid of a triangle are collinear, see [3] .

If we transform this set by the matrix $b_{ii} = a_i$ , we obtain the points $( a_1, a_2. \alpha_3 )$, $( s - a_1, s-a_2, s - a_3 )$, ( 1, 1, 1 ) . Thus, we have derived

**RESULT 1.** The Lemoine point, the middlespoint and the incentre of a tainngle are collinear .

**PROPOSITION 2.** The isogonal conjugate os the Gergonne point, and the isogonal conjugate of the Nagel point, and the incentre are collinear, see [2] .

Here we apply the collineation $b_{ii} = \frac{1}{a_i}$ and obtain the points

$$\left( \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \right), \left( s-a_1, s-a_2, s-a_3 \right), \left( \frac{1}{s-a_1}, \frac{1}{s-a_2}, \frac{1}{s-a_3} \right).$$

We now have

**RESULT 2.** The middlespoint of a given triangle is collinear with its isogonal conjugate and the centroid .

If we consider again the points of proposition 2 and apply the collineation $b_{ii} = \frac{1}{a_i^2}$ we obiain the points $\left( \frac{1}{a_1^2}, \frac{1}{a_2^2}, \frac{1}{a_3^2} \right)$,

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$$\left(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_3}\right), \left(\frac{1}{a_1(s-a_1)}, \frac{1}{a_2(s-a_2)}, \frac{1}{a_3(s-a_3)}\right),$$

thus we have

**RESULT 3.** The isotomic conjugate of the incentre, the Nagel point, and the Gergonne point are collinear.

**PROPOSITION 3.** The Gergonne point, the middlespoint, and the centroid of a triangle are collinear, see [1].

We now apply the same collineation as for result 2 and obtain the points

$$\left(\frac{1}{a_1^2(s-a_1)}, \frac{1}{a_2^2(s-a_2)}, \frac{1}{a_3^2(s-a_3)}\right), \left(\frac{s-a_1}{a_1}, \frac{s-a_2}{a_2}, \frac{s-a_3}{a_3}\right),$$

$$\left(\frac{1}{a_1^2}, \frac{1}{a_2^2}, \frac{1}{a_2^2}\right).$$ we can now state

**RESULT 4.** The isotomic conjugate of the middlespoint, the Nagel point, and the isotomic conjugate of the incentre are collinear.

We now summarize these results in Table 2. For brevity, the collineations are again represented by the general diagonal terms of the defining matrices.

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Table 2

| | Known Results | | Derived Results |
|---|---|---|---|
| 1. | $\{I, N, S\}$ | $\xrightarrow{a_i}$ | $\{K, M, I\}$ |
| 2. | $\{G^g, N^g, I\}$ | $\xrightarrow{\frac{1}{a}}$ | $\{M, M^g, S\}$ |
| 3. | $\{G^g, N^g, I\}$ | $\xrightarrow{\frac{1}{a_i^2}}$ | $\{I^t, N, G\}$ |
| 4, | $\{G, M, S\}$ | $\xrightarrow{\frac{1}{a_i}}$ | $\{M^t, N, I^t\}$ |

By examining the results in the table, we are able to state two further results. From the incident sets $\{M, M^g, S\}$ and $\{G, M, S\}$, we obtain

**RESULT 5.** The middlespoint of a triangle and its isogonal conjugate nre collinear with the Gergonne point and the centroid.

Similarly, thr sets $\{I^t, N, G\}$ and $\{M^t, N, I^t\}$ yield.

**RESULT 6.** The Nagel and Gergonne points of a triangle are collinear with the isotomic conjugates of the incentre and the middlespoint .

**4. Conclusion** As indicated earlier, material of this sort has a twofold value, Not only does it provide concrete examples that may be used in teaching the more abstract projective geometry but also, it helps

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to keep alive at least a portion of that all too quickly fading subject of Euclidean geometry. The material itself should fit the curriculum of most undergraduate courses in geometry and since, many other such relations no doubt exist, there is good potential here for student research projects. Since, in our view, there is no greater motivator than the joy of discovery, such activity is a necessary part of the teaching-learning process. As a concluding remark we note that while much has been written, so also much nemains to be discovered about that most fascinating configuration-the triangle .

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# THE *K*-TRANSFORMATION INDEX

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## ABSTRACT

In this paper we introduce the *K*-transformation index, defined by : $F_s(\mu) = \int_0^\infty f(t)\, K_\mu(st)\, dt,\ \mu = \sigma + i\tau,\ s \in C$ (1)

Two inversion theorems and an application of (1) on the theory of diffraction of transient electromagnetic waves are analysed.

## 1. INTRODUCTION

The Kontorovich-Lebedev transformation is defined by

$$F(\tau) = \int_0^\infty f(t)\ K_{i\tau}(t)\, dt,\ \tau > 0$$

where $K_\mu(t)$ is the modified Bessel function of third kind and order $\mu$.

The corresponding inversion formula, under certain conditions [5], is given by :

$$f(t) = \frac{2}{\pi^2 t} \int_0^\infty \tau \sinh \pi\tau\ K_{i\tau}(t)\ F(\tau)\, d\tau\ .$$

Later N. N. Lebedev [6] define a Kontorovich-Lebedev type integral transformation of a real valued function $f$ ( *I*-transformation index ) by :

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$$H_s(\mu) = \int_0^\infty f(t)\, I_\mu(st)\, dt,\ \mu = \sigma + i\alpha,\ s \in C, \tag{1.1}$$

where $I_\mu(t)$ is the modified Bessel function $f$ of first kind and order $\mu$. The inversion formula is [6] :

$$f(t) = \frac{1}{\pi i t} \int_{\sigma - i\infty}^{\sigma + i\infty} \mu\, H_s(\mu)\, K_\mu(st)\, d\mu,\ \sigma > 1 .$$

Now we define the $K$-transformation index by :

$$F_s(\mu) \int_0^\infty f(t)\, K_\mu(st)\, dt,\ \mu = \sigma + i\tau,\ s \in C. \tag{1.2}$$

The inversion method presented in [6] is utilised here to give an inversion formula for (1.2) :

$$f(t) = \frac{1}{\pi i t} \int_{\sigma - i\infty}^{\sigma + i\infty} \mu\, F_s(\mu)\, I_\mu(st)\, d\mu,\ \sigma > 1 .$$

The same inversion formula is proved by using the method of Green's functions developed in [3] .

Also we study a theorem of convergence and some operational rules for (1.1). Relations between (1.2) and Laplace and iterated Laplace transformations are presented in Propositions 2.2 and 2.3 respectively .

Finally, we solve a problem of diffraction of transient electromagnetic waves by a wedge ( *see* [2] ) by using the Laplace transform and our transformation .

## 2. Some properties of the $K$-transformation index .

We now establish some properties of the $K$-transformation index.

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Theorem 2.1 ( Convergence )

Let $F_s(\mu) = \int_0^\infty f(t)\, K_\mu(st)\, dt$, where $s, \mu \in C$ and $f(t)$ represents a function locally integrable on $(0, \infty)$ such that :

$$f(t) = O(t^\beta) \quad as\ t \to 0^+ \ (\beta \in R) \tag{2.1}$$

and $$f(t) = O(e^{\lambda t}),\ as\ t \to \infty\ (\lambda \in R,\ Re(s) > \lambda) \tag{2.2}$$

*If* $-1-\beta < Re(\mu) < 1+\beta$, *then* $F_s(\mu)$ is absolutelly convergent .

**Proof.** In effect, let $\varepsilon > 0$, $\varepsilon < T < \infty$ . We have :

$$\int_0^\infty |f(t)|\ |K_\mu(st)|\, dt = \int_0^\varepsilon |f(t|\ |K_\mu(st)|\, dt +$$

$$\int_\varepsilon^T |f(t)|\ |K_\mu(st)|\, dt + \int_T^\infty |f(t)|\ |K_\mu(st)|\, dt.$$ The second of integrals exists since $f$ is a locally integrble function on $(0,\infty)$ .

Now, $$\int_0^\varepsilon |f(t)|\ |K_\mu(st)|\, dt = \int_0^\varepsilon |f(t)|\ |t^{-\mu} - t^{\mu}|\, dt \leqslant$$

$$\leqslant \int_0 |f(t)|\, t^{-Re(\mu)}\, dt + \int_0^\varepsilon |f(t)|\, t^{Re(\mu)}\, dt.$$ By the hypothesis (2.1) and since $\beta \geqslant max\,(-1+Re(\mu), -1-Re(\mu))$, we can deduce that

$$\int_0^\infty |f(t)|\ |K_\mu(st)|\, dt$$ is convergent. Moreover, one has

$$\int_T^\infty |f(t)|\ |K_\mu(st)|\, dt \leqslant \int_T^\infty |f(t)|\, t^{-1/2}\, c^{-Re(s)\, t}\, dt .$$

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Hence, in virtue of (2.2) this integral is convergent .

The function $K_\mu(st)$ verifies the modified Bessel's differential equation : $t^2y'' + ty' - (\mu^2 + s^2t^2)\,y=0$ . Therfore,

$$A^k_{s,t}\,(K_\mu(st)) = \mu^{2k}(st), \text{ where } A_{s,t} = t^2D^2 + tD - s^2t^2 .$$

**Proposition 2.1**

*Let* $f \in C^2(0\ \infty)$, such that :

$$f(t) = O(t^\beta) \text{ as } t \longrightarrow 0^+, \tag{2 3}$$

$$f(t) = O(e^{\lambda t}) \text{ as } t \longrightarrow \infty, (\lambda \in R,\ Re(s) > \lambda)$$

*If* $-1-\beta < Re(\mu) < 1 + \beta$ :

$$\int_0^\infty A'_{s,t}(f(t))\,K_\mu(st)\,dt = \mu^2\int_0^\infty f(t)\,K_\mu(st)\,dt,$$

where $A'_{s,t}(f(t)) = t^2D^2f(t) + 3tDf(t) + (1 - s^2t^2)f(t)$

**Proof** . The conditions (2.3) imply the hypothesis of Theorem 2.1 and, therefore, $f$ is transformable. By using the rule of integration by parts we get :

$$\int_0^\infty A'_{s,t}(f(t))\,K_\mu(st)\,dt = \int_0^\infty f(t)\,A_{s,t}(K_\mu(st))\,dt$$

$$= \mu^2\int_0^\infty f(t)\,K_\mu(st)\,dt .$$

Next, we establish relations between the Laplace and iterated Laplace transforms and $K$-transformation index .

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**Proposition 2.2**

If $f(t)$ is a Laplace transformable function, we have :

$$L\{f(t)\} = (2/\pi)^{1/2} \int_0^\infty \sqrt{(st)}\ K_{1/2}(st)\, f(t)\, dt.$$

**Proof.** In effect, $e^{-st} = (2st/\pi)^{1/2}\ K_{1/2}(st)$ and therfore

$$\int_0^\infty f(t)\, e^{-st}\ dt = (2/\pi)^{1/2} \int_0^\infty (st)^{1/2}\ K_{1/2}(st)\, f(t)\, dt.$$

**Propositon 2.3**

Let $\mu \in C$ and $Re(s^2) > 0$ . Then : $\int_0^\infty f(t)\, K_\mu(st)\, dt =$

$$= \left(\frac{1}{2}\right)^{\mu+2} s^{-\mu} L\left\{ \tau^{-\mu-1} L\left\{ t^{(\mu-1)/2} f(\sqrt{t}) ; t \longrightarrow \frac{1}{4\tau} \right\} ; \tau \longrightarrow s^2 \right\}.$$

**Proof :** Since $\int_0^\infty f(t)\, K_\mu(st)\, dt = \left(\frac{1}{2}\right)^{\mu+1} \int_0^\infty f(t)\, (st)^\mu$

$$\int_0^\infty e^{-\tau - \frac{(st)^2}{4\tau}}\ \tau^{-\mu-1}\, d\tau dt.$$

By inverting to the order of integration and making $t^2 = h$,

$$\int_0^\infty f(t)\, K_\mu(st)\, dt = \left(\frac{1}{2}\right)^{\mu+2} s^\mu \int_0^\infty e^{-\tau}\, \tau^{-\mu-1} \int_0^\infty h^{\frac{\mu-1}{2}}\ e^{-\frac{s^2 h}{4\tau}}$$

$$f(\sqrt{h})\, dh d\tau.$$

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Now, making $\tau = s^2 x$, the last expression is equal to :

$$\left(\frac{1}{2}\right)^{\mu+2} s^{-\mu} \int_0^\infty e^{-s^2 x}\, x^{-\mu-1} \int_0^\infty h^{\frac{\mu-1}{2}}\, e^{-\frac{s^2 h}{4x}}\, f(\sqrt{h})\, dh\, dx =$$

$$= \left(\frac{1}{2}\right)^{\mu+2} s^{-\mu}\, L\left\{ \tau^{-\mu-1}\, L\left\{ t^{(\mu-1)/2} f(\sqrt{t}) ; t \longrightarrow \frac{1}{4\tau} \right\} ; \tau \longrightarrow s^2 \right\}.$$

From this relation, *if* $f(t) = t^\beta$, $\beta \in C$, $Re(s^2) > 0$, $Re\beta > |Re\mu| - 1$,

$$\int_0^\infty t^\beta K_\mu(st)\, dt = 2^{\beta-1} \frac{1}{s^{\beta+1}} \Gamma\left(\frac{\mu+\beta+1}{2}\right) \Gamma\left(\frac{\beta-\mu+1}{2}\right)$$

## 3. An inversion formula.

In this section we prove an inversion formula for (1.1).

The main result of this paper is :

**Theorem 3.1** ( Inversion formula )

*Let* $F_s(\mu) = \int_0^\infty f(t) K_\mu(st)\, dt$, $\mu = \sigma + i\tau$, $\sigma > 1$, where

$f(t)$ is bounded variation in each interval $(a, A)$, $0 < a < A$. (3.1)

and $f(t) K_\mu(st) \in L(0, \infty)$.

Then $$f(t) = \frac{1}{\pi i t} \int_{\sigma - i\infty}^{\sigma + i\infty} \mu\, F_s(\mu)\, I_\mu(st)\, d\mu$$

**Proof.** *Let* $J(t,T) = \frac{1}{\pi i t} \int_{\sigma - iT}^{\sigma + iT} \mu\, F_s(\mu)\, I_\mu(st)\, d\mu.$

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We will prove that

$$\lim_{T\to\infty} J(t, T) = f(t) .$$

Since $$J(t, T) = \int_0^\infty f(\zeta)\, d\xi \; \frac{1}{\pi i t} \int_{\sigma - iT}^{\sigma + iT} \mu\, K_\mu (s\xi)\, I_\mu (st)\, d\mu$$

and making the change of variables $\zeta = te^\theta$

$$J(t,T) = \int_{-\infty}^{\infty} f(te^\theta)\, e^\theta\, G_T(\theta)\, d\theta$$

where $$G_T(t) = \frac{1}{\pi i} \int_{\sigma - iT}^{\sigma + iT} \mu\, I_\mu (st)\, K_\mu (ste^\theta)\, d\mu .$$

Dixon and Ferrar [4] gave the integral representation :

$$K_\mu (ste^\theta)\; l_\mu (st) = \tfrac{1}{2} \int_\theta^\infty e^{-\mu\eta}\, J_0 (ste^{\theta/2}\, \Psi)\, d\eta , \qquad (3.2)$$

where $\Psi = (2ch\eta - 2ch\theta)^{1/2}$, $-\infty < \theta < \infty$, and $Re\mu > 1/4$ . Here $J_0$ is the Bessel function of order zero .

By using (3.2) and integrating by parts

$$G_T(\theta) = \frac{e^{-\sigma\theta}\, sen T\theta}{\pi\, \theta} + R_T(\theta) , \text{ where}$$

$$R_T(\theta) = -\frac{1}{\pi} \int_\theta^\infty e^{-\sigma\eta}\, \frac{sen\eta T}{\eta}\, \frac{\partial}{\partial\eta}\, J_0 (ste^{\theta/2}\, \Psi)\, d\eta .$$

Now, by Theorem of Dirichlet [ 1, p. 383 ]

$$\lim_{T\to\infty} \int_0^\infty f(te^\theta)\, e^{-\sigma\theta}\, \frac{sen T\theta}{\theta}\, d\theta = f(t)$$

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We have to see that

$$\lim_{T \to \infty} \int_{-\infty}^{\infty} f(te^{\theta}) R_T(\theta)\, d\theta = 0 .$$

Following once again some manipulations of Lebedev [6], one has

$$R_T(\theta) = \frac{-ste^{\theta/2}}{\pi T} \left\{ \tfrac{1}{2} st\, e^{-(\alpha+\frac{1}{2})\theta} \frac{sen h\theta}{\theta} \cos T\theta + \right.$$

$$+ \int_0^{\infty} e^{-\sigma\eta} \cosh\eta \cos T\eta \left[ \left( \frac{1}{\eta} - \frac{th\eta}{\eta^2} - \sigma \frac{th\eta}{\eta} \right) \frac{J_1(ste^{\theta/2}\Psi)}{\Psi} - \right.$$

$$\left. \left. - st\, e^{\theta/2} \frac{th\eta\, senh\eta}{\eta\, \Psi^2} J_2(ste^{\theta/2}\Psi) \right] d\eta \right\} .$$

Moreover, by virtue of well-known boundedness properties, we can prove for $\theta \geqslant 0$

$$|R_T|_{\theta \geqslant 0} \leqslant \frac{A_1 |s|^2 t^2 e^{-\sigma\theta}}{T} +$$

$$+ \frac{|s|\, te^{\theta/2}}{T} \int_{\theta}^{\infty} e^{-(\sigma-1)\eta} \left[ A_1 |s|\, te^{\theta/2} + A_3 |s|\, te^{\theta/2} + A_4 |s|^2 \right.$$

$$\left. t^3 e^{\theta/2} \right] d\eta$$

$$= \left[ \frac{A_1 |s|^2 t^2}{T} + \frac{A_5 |s|^2 t^2 e^{\frac{3}{2}\theta}}{T} + \frac{A_\sigma |s|^4 t^4 e^{\frac{3}{2}\theta}}{T} \right] e^{-\sigma\theta},$$

being $A_i$, $i = 1, \ldots, \sigma$ constants, and provided that $\sigma > 1$.

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On the other hand, for $\theta \leqslant 0$

$$R_{T/\theta \leqslant 0} = R_1 + R_2 \text{ , where}$$

$$R_1 = \frac{-ste^{\theta/2}}{\pi T} \left\{ \tfrac{1}{2} st\ e^{-(\sigma+\frac{1}{2})\theta} \frac{senh\theta}{\theta}\ cosT\theta + \right.$$

$$+\int_{-\theta}^{\infty} e^{-\sigma\eta}\ cosh\eta \left[ \left( \frac{1}{\eta} - \frac{th\eta}{\eta^2} - \sigma \frac{th\eta}{\eta} \right) \frac{J_1(ste^{\theta/2}\Psi)}{\Psi} - \right.$$

$$\left. \left. - st\ e^{\theta/2} \frac{th\eta}{\eta}\ \frac{senh\eta}{\Psi^2}\ J_2(ste^{\theta/2}\Psi) \right] \right.$$

and

$$R_2 = \frac{-ste^{\theta/2}}{\pi T} \int_{\theta}^{-\theta} e^{-\sigma\eta}\ cosh\eta\ cosT\eta \left[ \left( \frac{1}{\eta} - \frac{th\eta}{\eta^2} - \sigma \frac{th\eta}{\eta} \right) \frac{I_1(ste^{\theta/2}\Psi_1)}{\Psi_1} \right.$$

$$\left. - st\ e^{\theta/2} \frac{th\eta}{\eta}\ \frac{senh\eta}{\Psi_1^2}\ I_2(ste^{\theta/2}\Psi_1) \right] d\eta$$

*being* $\Psi = i\Psi_1$ , and $\Psi_1 = \left( 2cosh\theta - 2cosh\eta \right)^{1/2}$.

In effect, by using boundedness properties of the Bessel functions and integrating, we get

$$|R_1| \leqslant \frac{B_1 |s|^2 t^2 e^{-\sigma\theta}}{T} + \frac{|s|\ te^{\theta/2}}{T} \int_{-\theta}^{\infty} e^{-(\sigma-1)\eta}$$

$$\left[ B_2 |s|\ te^{\theta/2} + B_3 |s|\ te^{\theta/2} + B^4 |s|^3 t^3 e^{\theta/2} \right] d\eta =$$

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$$= \frac{B_1 \mid s \mid^2 t^2 e^{-\sigma\theta}}{T} + \frac{B_5 \mid s \mid^2 t^2 e^{\sigma\theta}}{T} + \frac{B_\sigma \mid s \mid^4 t^4 e^{\sigma\theta}}{T} -$$

provided that $\sigma > 1$, and where $B_i$, $i = 1, \ldots, 6$, are constants.

Also,

$$\mid R_2 \mid \leqslant \frac{A_7 \mid s \mid te^{\theta/2}}{T} \int_0^{-\theta} e^{(\sigma+1)\eta} \frac{\mid I_1 ( ste^{\theta/2} \Psi_1 ) \mid}{\Psi_1} \, d\eta +$$

$$+ \frac{A_8 \mid s \mid^2 t^2 e^{\theta}}{T} \int_0^{-\theta} e^{(\sigma+1)\eta} \frac{senh\eta}{\Psi_1^2} \mid I_2 ( ste^{\theta/2} \Psi_1 ) \mid d\eta = Q_1 + Q_2$$

where

$$Q_1 = \frac{A_7 \mid s \mid te^{\theta/2}}{T} \int_0^{\varepsilon} e^{(\sigma+1)\eta} \frac{\mid I_1 (ste^{\theta/2} \Psi_1) \mid}{\Psi_1} \, d\eta +$$

$$+ \frac{A_7 \mid s \mid te^{\theta/2}}{T} \int_{\varepsilon}^{\theta} e^{(\sigma+1)\eta} \frac{\mid I_1 (ste^{\theta/2} \Psi_1) \mid}{\Psi_1} \, d\eta$$

and

$$Q_2 = \frac{A_8 \mid s \mid te^{\theta/2}}{T} \left[ \frac{\mid I_1 ( 2ste^{\theta/2} senh\theta/2 ) \mid}{2senh\theta/2} - \frac{1}{2} \mid s \mid te^{(\sigma+3/2)\theta} \right] + A_9 Q_1$$

being $A_7$, $A_8$ and $A_9$ suitable positive constants, Moreover, the following estimates hold :

$$\frac{A_7 \mid s \mid te^{\theta/2}}{T} \int_0^{\varepsilon} e^{(\sigma+1)\eta} \frac{\mid I_1 ( ste^{\theta/2} \Psi_1 ) \mid}{\Psi_1} \, d\eta$$

$$\leqslant \frac{A_7 \mid s \mid te^{\theta/2} \mid I_1 (2ste^{\theta/2} senh\theta/2 ) \mid e^{(\sigma+1)\varepsilon} - 1}{T ( \sigma+1) \, 2senh\theta/2}$$

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$$\frac{A_7 \mid s \mid te^{\theta/2}}{T}\int_{\varepsilon}^{-\theta} e^{(\sigma+1)\eta}\,\frac{\mid I_1\,(ste^{\theta/2}\,\Psi_1)\mid}{\Psi_1}\,d\eta \leqslant$$

$$\leqslant \frac{A_{10} \mid s \mid t}{T}\,cosh^{\sigma+1}\theta\int_0^{\pi/2} cos^{2\sigma+1}\,x \mid I_1(\,st\,(2e^{\theta}\,cosh\theta)\,senx\,)\mid dx,$$

( $A_{10} =$ *constant* ) and since

$$\int_0^{\pi(2} cos^{2\sigma+1}\,x\;I_1\;(\,z\,senx\,)\,dx = \tfrac{1}{2}\left[\frac{I_\sigma\,(z)\;\Gamma\,(z+1)}{(\,z/2\,)^{\sigma}} - 1\right],\ \text{therefore}$$

$$\frac{A_7 \mid s \mid te^{\theta/2}}{T}\int_{\varepsilon}^{-\theta} e^{(\sigma+1)\eta}\;\mid I_1\,(\,ste^{\theta/2}\,\Psi_1\,)\mid\;\frac{}{\Psi_1}\;d\eta \leqslant$$

$$\leqslant \frac{A_{11}\,cosh^{\sigma+1}\,\theta}{T \mid st \mid^{\sigma}\,(\,2e^{\theta}\,cosh\theta\,)^{\frac{\sigma+1}{2}}}\left[\mid I_\sigma\,(\,st\,(2e^{\theta}\,cosh\theta)\,)\mid -\right.$$

$$\left.\frac{\mid s \mid^{\sigma}\,(\,2e^{\theta}\,cosh\theta\,)^{\sigma/2}}{2^{\sigma}\;\Gamma\,(\sigma+1)}\right]$$

where $A_{11}$ is a constant. Therefore, we can obtain

$$\mid R_T \mid_{\theta} \leqslant \frac{M\,(\,t,\,\theta\,)}{T}$$

where $M\,(\,t,\theta\,)$ is a continuous function of two variables $t$, $\theta$, and hence

$$\lim_{T\to\infty}\int_0^{\infty} f\,(\,te^{\theta}\,)\;e^{\theta}\;R_T\,(\theta)\,d\theta = 0$$

## 4. Inversion by Green's functions,

The modified Bessel's differential equation is

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$$t^2 y'' + ty' - (s^2 t^2 + \mu^2) y = 0 \tag{4.1}$$

The delta function $\delta(r - \rho)$ can be written as

$$t\,\delta(t - \rho) = -\frac{1}{2\pi i} \oint G(t, \rho, \lambda)\, d\lambda \tag{4.2}$$

where $G(t, \rho, \lambda)$ is the Green's function of the operator in (4.1), satisfying the equation

$$\left[D^2 + \frac{1}{t} D - \left(s^2 + \frac{\mu^2}{t^2}\right)\right] G = -\delta(t - \rho)$$

and the integration is over a suitable contour in the $\lambda$-plane including all the singularities of the Green's function in tesms of $\lambda$.

If $Re(\mu) > 0$ and since $I_\mu(st)$ and $K_\mu(st)$ are independent solutions of (4.1) verifying

$$R_\mu(st) \to 0,\ t \to \infty,$$

$I_\mu(st)$ is finite as $t \to 0$, then the Green's function $G(t, \rho, \lambda)$ can be represented as

$$G(t, \rho, \lambda,) = \begin{cases} \dfrac{I_\mu(st)\, K_\mu(s\rho)}{Wt}, & t < \rho \\ \dfrac{I_\mu(s\rho)\, K_\mu(st)}{Wt}, & t > \rho \end{cases} \tag{4.3}$$

where $W$ is the wronskian $W(I_\mu, K_\mu)$ and is a function of $\mu$ and $t$. The form of the two expresslons of (4.3) shows that the same path of integration in (4.2) holds for $t > \rho$ and $\rho > t$. Now, the function $f(t)$, defined on $0 \leq t < \infty$ can be written as

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$$f(t) = \int_0^\infty f(\rho)\ \delta(t-\rho)\, d\rho$$

Therefore by (4.3) and (4.2) we get

$$f(t) = -\frac{1}{2\pi i}\int_0^\infty f(\rho)\left\{ \oint \frac{K_\mu(s\rho)\, I_\mu(st)}{W t^2}\ d\lambda \right\} d\rho$$

since

$$\oint \frac{K_\mu(s\rho)\, I_\mu(st) - I_\mu(s\rho)\, K_\mu(st)}{W t}\ d\lambda = 0,$$

for want of singularity. Interchanging the orders of integration (assuming it to be possible) and being $\lambda = \mu^2$

$$f(t) = -\frac{1}{2\pi i}\oint 2\mu \frac{I_\mu(st)}{W t^2}\ d\mu \int_0^\infty f(\rho)\ K_\mu(s\rho)\, d\rho .$$

Using the formula $W(I_\mu, K_\mu) = -1/t$,

$$f(t) = \frac{1}{\pi i t}\oint \mu\, I_\mu(st)\, d\mu \int_0^\infty f(\rho)\, K_\mu(s\rho)\, d\rho$$

where the path of integration in some abscissae $\sigma \geqslant 0$ in the plane of the variable $\mu = \sigma + i\tau$, since the singularities of $I_\mu$ are at $\mu = 0, -1, -2, \ldots$, therefore

$$f(t) = \frac{1}{\pi i t}\int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, I_\mu(st\, d\mu \int_0^\infty f(\rho)\, K_\mu(s\rho)\, d\rho$$

## 5. A Parseval relation for the *K*-transformation index.

Let $F_s(\mu) = \int_0^\infty f(t)\, K_\mu(st)\, dt$

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$$G_s(\mu) = \int_0^\infty g(t)\, K_\mu(st)\, dt,$$

the following Parseval relation holds :

$$\frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{a+i\infty} \mu\, sen\mu\pi\, F_s(\mu)\, G_s(\mu)\, d\mu = \int_0^\infty f(t)\, G(t)\, dt$$

In effect,

$$\frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, sen\mu\pi\, F_s(\mu)\, G_s(\mu)\, d\mu =$$

$$= \int_0^\infty g(t)\, \frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, sen\mu\pi\, F_s(\mu) \frac{\pi}{2\, sen\mu\pi} [\, I_{-\mu}(st) - I_\mu(st)\,]\, d\mu\, dt$$

$$= \int_0^\infty g(t)\, \frac{i}{2\pi t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, F_s(\mu)\, [\, I_{-\mu}(st) - I_\mu(st)\,]\, d\mu\, dt$$

Now

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, F_s(\mu)\, I_{-\mu}(st)\, d\mu = -\int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, F_s(\mu)\, I_\mu(st)\, d\mu,$$

*since* $F_s(\mu) = F_s(-\mu)$.

$$\frac{i}{\pi^2 t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, sen\mu\pi\, F_s(\mu)\, G_s(\mu)\, d\mu =$$

$$= \int_0^\infty g(t) \left(\frac{-i}{\pi t}\right) \int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, F_s(\mu)\, I_\mu(st)\, d\mu\, dt =$$

$$= \int_0^\infty g(t)\, \frac{1}{\pi i t} \int_{\sigma-i\infty}^{\sigma+i\infty} \mu\, F_s(\mu)\, I_\mu(st)\, d\mu\, dt = \int_0^\infty f(t)\, g(t)\, dt\,.$$

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## 6. An operational calculus.

Let $P$ be any polynomial. Consider the differential equation

$$P(A'_{s,t})f = g \tag{6.1}$$

where $g$ is a given transformable function. The $K$-transformation index generates an operational calculus through which a solution $u$ for (6.1) can be found. By applying this transform to (6.1) and invoking the Proposition 2.1 we obtain

$$F(\mu) = \frac{G(\mu)}{P(\mu^2)}$$

We may apply the inversion formula to determine $f$, solution to (6.1). The extension of this technique to simultaneous differential equations of the form (6.1) is straightforward.

## 7. An application of the $K$-transformation index.

The problem to which the $K$-transformation index is applied is the one considered in [2].

An infinitely long line source of uniformly distributed current, whose density is a function of time alone, lies parallel to the edge of a wedge with perfectely conducting walls in a region of conductivity $\sigma$, dielectric constant $\varepsilon$ and permeability $\lambda$. Cylindrical coordinates $(r, \theta, z)$ with the tip of the wedge coincident with the $z$-axis are used. The angle of the wedge is taken to be $\alpha$ and the line source passes through $(r_0, \theta_0)$, $0 \leqslant \theta_0 \leqslant \alpha$. Since the problem is two dimensional. Maxwell's equations (in m. k. s. units) for the region within the wedge are

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$$a)\ \frac{1}{r}\frac{\partial}{\partial r}(rH_\theta) - \frac{1}{r}\frac{\partial H_r}{\partial \theta} = (\sigma+\varepsilon\frac{\partial}{\partial t})E_z + M(t)\frac{\delta(r-r_0)}{r}\delta(\theta-\theta_0)$$

$$b)\ \frac{1}{r}\frac{\partial E_z}{\partial \theta} = -\lambda\frac{\partial H_r}{\partial t},\quad c)\ \frac{\partial E_z}{\partial r} = \lambda\frac{\partial H_\theta}{\partial t} \qquad (7.1)$$

where $r^{-1} M(t)\,\delta(r-r_0)\,\delta(\theta-\theta_0)$ represents a current density corresponding to a line source passing through $(r_0, \theta_0)$. $M(t)$ is a function of time alone and $\delta(x)$ denotes the Dirac delta function. The conditions on the boundary are

$$E_z = 0 \text{ on } \theta = 0, \alpha \qquad (7.2)$$

$$H_r = H_\theta = E_z = 0 \qquad \text{at } t = 0 \text{ and as } r \to \infty$$

A Laplace transform in $t$ and the $K$-transformation index in $r$, with $s^2 = \mu\eta(\sigma+\varepsilon\eta)$, are used to reduce these equations to an ordinary differential equation in $\theta$.

Multiplying equation 7.1a) by $r K_\mu(sr) e^{-\eta t}$, 7.1b) by $K_\mu(sr)e^{-\eta t}$, 7.1c) by $r\frac{\partial}{\partial r}K_\mu(sr)e^{-\eta t}$ and integrating with respect to $t$ and $r$ over the range $(0,\infty)$ we find

$$-H_\theta - \frac{dH_r}{d\theta} = (\sigma+\varepsilon\eta)\,\xi_z^{(1)} + \bar{M}(\eta)K_\mu(sr_0)\delta(\theta-\theta_0)$$

$$\frac{d\xi_z}{d\theta} = -\lambda\eta H_r \qquad (7.3)$$

$$-s^2\,\xi_z^{(1)} - \lambda^2\xi_z = \lambda\eta H_\theta$$

where

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$$\xi_z(\mu, \theta, \eta) = \int_0^\infty \frac{\bar{E}_z(r, \theta, \eta)}{r} K_\mu(sr)\, dr =$$

$$= \int_0^\infty \frac{K_\mu(sr)}{r}\, dr \int_0^\infty E_z(r, \theta, t)\, e^{-\eta t}\, dt$$

$$\xi_z^{(1)}(\mu, \theta, \eta) = \int_0^\infty \bar{E}_z\, K_\mu(sr)\, r\, dr,$$

$$H_\theta(\mu, \theta, \eta) = \int_0^\infty \bar{H}_\theta\, r\, \frac{\partial K_\mu(sr)}{\partial r}\, dr$$

$$H_r(\mu, \theta, \eta) = \int_0^\infty \bar{H}_r\, K_\mu(sr)\, dr$$

and the bar above a component denotes its Lplace transform with respect to $t$.

From equations (7.3) we find that the equation for $\xi_z$ is

$$\left(\frac{d^2}{d\theta^2} + \mu^2\right) \xi_z = \lambda\eta\, \bar{M}(\eta)\, K_\mu(sr_0)\, \delta(\theta - \theta_0) \qquad (7.4)$$

The solution of equation (7.4) subject to the boundary conditions (7.2) can be written as

$$\xi_z = \lambda\eta\, \bar{M}(\eta)\, \frac{sen\mu(\alpha - \theta_0)\, sen\mu\theta}{\mu\, sen\mu\alpha}\, K_\mu(sr_0), \qquad 0 \leqslant \theta < \theta_0$$

When $\theta > \theta_0$ the positions of $\theta$ and $\theta_0$ are interchanged. Inverting with respect to the $K$-transformation index we have ($\theta < \theta_0$)

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$$E_z\ (r, \theta, t) = \frac{\lambda}{2\pi^2} \int_{c-i\infty}^{c+i\infty} \eta\, \bar{M}\ (\eta)\, e^{\eta t}\, d\eta\ .$$

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{sen\mu\ (\alpha-\theta_0)\ sen\mu\theta}{sen\mu\alpha}\ K_\mu\ (sr_0)\ I_\mu\ (sr)\ d\mu$$

**Final Observation .**

We also note that the *K*-transformation index may be extended to distributions along the lines of A. H. Zemanian [ 12 ]. This will require the construction of a Fréchet space that contains the kernel $K_\mu\ (st)$ as an element . The transform will then be defined on the dual space as the application of the distribution to this kernel . This will be the subject of a later paper .

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# APPLICATIONS OF FIXED POINTS TO APPROXIMATION THEORY

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Several interesting results have been given in Approximation theory using the fixed point theory . A very good source of such illustrations may be found in a paper due to Chener [2] . For a detailed survey paper one is referred to [5] .

The work of Brosowski [1], Singh [6], and Hicks and Humphries [4] involves nonexpansive mappings . Recall that if $f : X \to X$, where $X$ is a normed linear space, and $\|fx - fy\| \leqslant \|x - y\|$ for $x, y \in X$, then $f$ is said to be nonexpansive map

The aim of this paper is to prove results in this direction and derive known results as corollaries .

We recall that, given $x \in X$ and $C \subset X$, the set of the best $C$-approximants $x$ consist of all $y \in C$ such that $\|y-x\| = inf \{ \|z-x\|, z \in C \}$ .

The main theorem is given below .

**Theorem 1.** Let $C$ be a subset of a Banach space $X$ and $T : X \to X$ be a mapping. Let $T(\partial C) \subset C$ ( $\partial C$ stands for the boundary of $C$ ), $Tx_0 = x_0$, $x_0 \in X$ and $T$ satisfy the following condition :

$$\|Tx - Ty\| \leqslant \alpha\|x - y\| + \beta ( \|x - Tx\| + \|y - Ty\| ) + \gamma ( \|x - Ty\| +$$

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$\| y - Tx \|$ ) for all $x, y \in X$, where $\alpha, \beta$ and $\gamma$ are reals with $\alpha + 2\beta + 2\gamma \leqslant 1$.

Further, if the set $D$ : the set of best $C$-approximants to $x_0$, is non-empty compact convex and $T$ is continuous on $D$, then $T$ has a fixed point in $D$ closest to $x_0$.

**Remarks :** In [1] a similar result is proved for a linear, nonexpansive map $T$ from $C$ to itself, and an extended version in [6] where $T : C \to C$ is taken. Here we do not require linearity, nonexpansiveness of $T$ on $X$ and $T : C \to C$ not needed, only $T(\partial C) \subset C$ serves the purpose.

**Proof :** First we show that

$\| Tx - Tx_0 \| \leqslant \| x - x_0 \|$ for all $x \in X$.

Clearly,

$$\| Tx - Tx_0 \| \leqslant \alpha \| x - x_0 \| + \beta ( \| x - Tx \| + \| x_0 - Tx_0 \| ) + \gamma - ( ( \| x Tx_0 \| + \| x_0 - Tx \| ).$$

Since $Tx_0 = x_0$, We get

$$\| Tx - Tx_0 \| \leqslant \alpha \| x - x_0 \| + \beta \| x - Tx \| + \gamma ( \| x - x_0 \| + \| Tx_0 - Tx \| ).$$

Now, $\| x - Tx \| \leqslant \| x - x_0 \| + \| Tx - Tx_0 \|$, therefore we have

$$\| Tx - Tx_0 \| \leqslant \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \| x - x_0 \|,$$

*i. e,* $\| Tx - Tx_0 \| \leqslant \| x - x_0 \|$ for all $x \in X$ (1).

We claim that $T : D \to D$.

Let $y \in D$. Then

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$\| Ty - x_0 \| = \| Ty - Tx_0 \| \leqslant \| y - x_0 \|$ by (1).

Clearly $y$ cannot be in $C$, hence $y \in \partial C$ ( see *e. g.* [4], Lemma 1 ). Since $y$ is nearest to $x_0$ and $T(\partial C) \subset C$, so Ty is also nearest to $x_0$, *i. e.* $Ty \in D$.

Since $D$ is a non-empty compact convex subset of $X$ and $T : D \to D$ is a continuous map therefore $T$ has a fixed point in $D$ by Schauder fixed point theorem. So $Tx = x$ say, where $x \in D$ closest to $x_0$.

**Theorem 2.** Let $T : X \to X$ be a mapping on Banach space $X$ satisfying the following :

I ) $Tx_0 = x_0$, $x_0 \in X$,

II ) $T(\partial C) \subset C$, where $C$ is a subset of $X$,

III ) the set $D$ : the set of best $C$-approximants to $x_0$ is compact and starshaped,

IV ) $T$ is nonexpansive on $D$ and $\| Tx - Tx_0 \| \leqslant \| x - x_0 \|$ for all $x \in D$. Then $T$ has a fixed point in $D$ closest to $x_0$.

Recall that a set $B$ is said to be starshaped with star center $p \in B$ if $\lambda x + (1 - \lambda) p \in B$, $0 \leqslant \lambda \leqslant 1$ for all $x \in B$. A convex set is always starshaped, but not conversely.

Singh [7] proved a similar theorem under the condition $T : C \to C$. It is shown in [4] that $T(\partial C) \subset C$ suffices.

Our proof follows similar lines as in [6].

**Proof :** As in Theorem 1 it is easy to show that $T : D \to D$.

Since $T$ is a nonexpansive mapping on $D$ and $D$ is strashaped wtth star center $p$ therefore we define $Tk_n x = k_n Tx + (1 - k_n) p$ for all $x \in D$, where $\{k_n\}$ is a sequenee of real numbers, $0 \leqslant k_n < 1$, converges

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to 1 as $n \to \infty$. Now, $T_n : D \to D$ for each $n$, since so is $T$ ( We write $n$ in place of $k_n$ ).

Also, $\| T_n x - T_n y \| = k_n \| Tx - Ty \| \leqslant k_n \| x - y \|$. Since $T$ is non expansive on $D$ aad $0 \leqslant k_n < 1$, then

$$\| T_n x - T_n y \| < \| x - y \|, \quad x \neq y \text{ for all } x, y \in D.$$

Then each $T_n$ has a unique fixed point say $y_n$, for $n = 1,2, \ldots$ ( see Edelstein [3] ), *i e.*

$$T_n y_n = y_n, \; n = 1, 2, 3, \ldots$$

Since $D$ is compact therefore $\{y_n\}$ has a subsequence $\{y_{n_i}\}$ converging to $y$ say .

We claim that $Ty = y$ in $D$.

Since

$$y_{n_i} = T_{n_i} y_{n_i} = (1 - k_{n_i}) p + k_{n_i} T y_{n_i},$$

taking limit as $i \to \infty$ and using the fact that $T$ is continuous we get that

$$y = Ty \text{ since } k_{n_i} \to 1.$$

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# A REMARK ON A SMOOTHED FOURIER SERIES *

*By*

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*( Received : November 28, 1988 )*

Many recent papers studied variaus problems connected to the approximation of a given function $f$, defined on the $N$-dimensional torus $T^N$, by its Fourier polynomials, smoothed by multiplying every coefficient f $(n)$ by a factor of the form $[\,1+\sigma P(n)\,]^{-1}$, where $P$ is a homogeneous polynomial of even degree $K$ and $\sigma$ is a positive real parameter. See [1], [2], [4], [5, .

In particular if $f \in C\,(\,T^N\,)$, ( the space of continuous functions with the supremum norm ) it is known ( [1] *Th.* 1 ) that

$$f_\sigma(t) \sim \sum_{n \in Z^N} \frac{f(n)}{1+\sigma P(n)}\, e^{2\pi i n t} \qquad , t \in T^N$$

( $Z^N$ the lattice of integer points of $R^N$ ), is a function of $C\,(\,T^N\,)$ and

$$\| f - f_\sigma \|_\infty \to 0 \qquad \text{when } \sigma \to 0^+$$

Moreover if f belongs to the Lipschitz Lip $(\alpha, C\,(\,T^N\,))$, $0 < \alpha \leqslant 1$, and $c$ is a suitable positve constant ( [2], *Th.* 4 )

$$\| f - f_\sigma \|_\infty \leqslant c\, \sigma^{\alpha/k}$$

and, if $N$ is odd, the above approximation is sharp, at least when $P$ is

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radial [5].

In this paper we show that a similar result still holds when $N=2$ and $K=4$. Indeed we prove the following

**Theorem** . *Let* $N=2$ and $P$ the radial polynomial of degree $K=4$, *i, e* $P(x)=|x|^4$.

Then there exists $M>0$ such that

$$\sup_{f \epsilon Lip(\alpha, C(T^2))} \| f - f_\sigma \|_\infty > M \sigma^{\alpha/k}$$

**Remark** . This theorem and the one proved in [5] guarantee that the given approximation is the best one both in every odd dimension $N$ and in a particular case when $N=2$. These results suggest that the same holds in every dimension, althogh it seems that the demonstrative techniques adopted can not be extended to the general case .

**Proof** : From [1] *Th.* 5 we know that given $g : R^N \to R$ defined by $g(x) = [1 + P(x)]^{-1}$, if we call $G$ the function belonging to $L_1(R^N)$ such that $G(x) = g(x) \; \forall x \epsilon R^N$ and define :

$$G_\sigma(x) = \sigma^{-N/K} G(x\sigma^{-1/k})$$

and $$K_\sigma(t) = \sum_{n \epsilon Z^N} G_\sigma(t+n) \qquad , t \epsilon T^N$$

then we have

$$K_\sigma(t) \sim \sum_{n \epsilon Z^N} \frac{1}{1+\sigma P(n)} e^{2\pi i n t} \quad \text{and} \quad f_\sigma = f * K_\sigma.$$

We also recall that, by hypothesis, $g \epsilon L_1(R^2)$ and so, by the inversion formula, $g(x) = G(x)$. Being $g$ radial also $G$ is and it can be computed, ( [6] *p.* 155 ) by the Hankel's transform

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$$G(x)=G(|x|)=2\pi\int_0^{+\infty} g(|x|)J_0(2\pi|x||y|)|x|d|x|.$$

By [3] Vol. 11, *p*. 7 we get

$$G(x) = -2\pi\, Kei_0(2\pi|x|)$$

where $Kei_0$ is one of the Kelvin functions, related to the modified Bessel function $K_\nu$ by

$$Kei_0(x) = (2i)^{-1}[K_0(xe^{i\pi/4}) - K_0(xe^{-i\pi/4})].$$

For every $\alpha\in(0,1]$ let us define

$$f_\alpha(x) = \begin{cases} |x|^\alpha & |x|\leqslant 1/2 \\ (1-|x|)^\alpha & 1/2 < |x| < 1 \end{cases}$$

The function $f_\alpha$ restricted to the fundamental domain

$$Q_2 = \{x\in R^2 : -1/2\leqslant x_j < 1/2,\ j=1,2\}$$

defines a continuous function ; furthermore $f_\alpha\in Lip(\alpha, C(T^2))$,

Define $f_{\alpha,\sigma} = K_\sigma * f_\alpha$ and continue $f_\alpha$ periodically in each variable throughout the whole space $R^2$. Then

$$\|f_{\alpha,\sigma} - f_\alpha\|_\infty \geqslant |f_{\alpha,\sigma}(0)| =$$

$$= \left|\int_{Q_2} K_\sigma(-x) f_\alpha(x)\,dx\right| = \left|\sum_{n\in Z^2}\int_{Q_2+n} G_\sigma(x+n) f_\alpha(x)\,dx\right| =$$

$$= \left|\int_{R^2} G_\sigma(x) f_\alpha(x)\,dx\right| = \left|\int_{R^2} G(x) f_\alpha(\sigma^{1/k}x)\,dx\right|.$$

If we consider $g$ defined as above for some positive constant $c$ we have

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$$\| f_{\alpha,\sigma} - f_\alpha \|_\infty \geqslant c \cdot \left| \int_0^{+\infty} G(|x|) f_\alpha(\sigma^{1/k}|x|)|x| d|x| \right| \geqslant$$

$$\geqslant c\ \sigma^{\alpha/k} \left| \int_0^{1/2\sigma^{-1/k}} G(|x|)|x|^{1+\alpha} d|x| \right| - c \left| \int_{1/2\sigma^{-1/k}}^{+\infty} |x| G(|x|) \right.$$

$$\left. f_\alpha(\sigma^{1/k}|x|) d|x| \right| = c\ \sigma^{\alpha/k} I_1(\sigma) - c\ I_2(\sigma).$$

Let us name :

$$I = \int_0^{+\infty} |x|^{1+\alpha} G(|x|) d|x|.$$

From [3, Vol. 1, p 391], we get

$$I = 1/2\ \pi^{-(1+\alpha)}\ \Gamma^2(1+\alpha/2)\ cos\ (\alpha\pi/4).$$

Since $I$ is positive for every $\alpha \in (0, 1]$, we can choose $\delta$ sufficiently small such that there *exists* $\sigma_0 = \sigma_0(\delta)$ in order to have

$$I_1(\sigma) > \delta \qquad \forall\ \sigma < \sigma_0(\delta).$$

On the other hand recalling that

$$K_1(y) \sim \left(\frac{\pi}{2y}\right)^{1/2} e^{-y} \qquad \text{when } y \to \infty$$

wr can say that

$$I_2(\sigma) < c\ \sigma^{-1/2k}\ e^{-3/2}\ \sigma^{-1/k} = 0(\sigma^\lambda)\ \forall\ \lambda > 0 \text{ whdn } \sigma \to 0^+$$

The proof is so complete.

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# SLIP FLOW IN A ROTATING CHANNEL

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## ABSTRACT

An attempt has been made to study the effects of the slip parameter on the flow formed when a staright channel bounded by two parallel walls which liquid is flowing under a constant pressure gradient is rotated about an axis perpendicular to the walls. A method for setting up an experiment to test the theoretical conclusions of the paper has been suggested .

## 1. INTRODUCTION

Greenspan [1] presented a detailed account of motion of rotating liquids enclosed within a body or vice-versa, including his own general theory of a contained rotating liquid. Rott and Lewellen [2] discussed the problems relating to the boundary layers and their interactions in rotating flows, nnd gave many examples of such inieractions . Another example of such interaction is the flow of a liquid under a constant pressure gradient between two rotating parallel walls, considered by Vidyanidhi and Nigam [6]. Accounting for the first order slip as in the works of Souudalgekar [4], and Sastry and Bhadram [3], this latter problem has been dealt with here .

## 2. BASIC EQUATIONS AND SOLUTION

The equations of motion and continuity for the steady state in

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a rotating frame of reference $O'X\,Y'Z'$ as given by Squire [5] for an incompressible liquid are

$$\rho(\vec{u'} \cdot \nabla')\vec{u'} - 2\rho(\vec{u'} \times \vec{\Omega'}) = -\nabla'\pi' + \rho\nu\nabla'^2\vec{u'} \quad (1)$$

$$\nabla' \cdot \vec{u'} = 0 \quad (2)$$

where

$$\pi' = P' - \tfrac{1}{2}\rho\,|\,\vec{\Omega'} \times \vec{r'}\,|^2 \text{ and } \vec{u'}, \vec{\Omega'}$$

and $\vec{r'}$ are the velocity vector, angular velocity, and the position vector, respectlvely .

We choose a right-handed Cartesian system such that the $Z'$-axis is perpendicular to the motion of the liquid flowing under the action of a constant pressure gradient $-\frac{\partial\pi'}{\partial x'}$ in the direction of $X'$-axis between two parallel walls $Z' = \pm L$

( Stationary relative to $O\,X'Y'Z'$ ) .

Assuming

$$\vec{u'} = \left\{ u'_{x'}(z'), u'_{y'}(z'), 0 \right\} \quad (3)$$

$$\vec{\Omega'} = (0, 0, \Omega') \quad (4)$$

and introducing non-dimensional quantities

$$r' = rL,\ u' = \frac{uQ}{\rho L}\ , \quad (5)$$

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$$\mathcal{R}' = \frac{\mathcal{R} Q}{\rho L^2} \quad , \quad \pi' = \frac{\pi Q^2}{\rho L^2}$$

where $Q$ is the mass flux given by

$$Q = ( Q'^2_{x'} + Q'^2_{y'} )^{\frac{1}{2}}$$

$$Q'_{x'} = \int_{-L}^{L} \rho\, u'_{x'}\, dz', \quad Q'_{y'} = \int_{-L}^{L} \rho\, u'_{y'}\, dz',$$

the equation (1) reduces to

$$-2u_y = P + \frac{1}{\alpha^2} \frac{d^2 u_x}{dz^2} \tag{6}$$

$$2u_x = \frac{1}{\alpha^2} \frac{d^2 u_y}{dz^2} \tag{7}$$

where

$$\alpha^2 = \frac{\mathcal{R}'}{\nu} L^2$$

$$P = - \frac{L}{Q \mathcal{R}'} \frac{\partial \pi'}{\partial x'} \tag{8}$$

We seek the solution of (6) and (7) subject to the boundary conditions

$$\left.\begin{aligned} u_x &= \beta \frac{\partial u_x}{\partial z} \\ u_y &= \beta \frac{\partial u_y}{\partial z} \end{aligned}\right\} \quad at\ z = \pm 1 \tag{9}$$

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where $\beta$ denotes non-dimensional first-order velocity slip as in the works of Soundalgekar [4], and Sastry Bhadram [3].

The solution of the above boundary-value problem is given by

$$\frac{u_x}{P_0} = [\ \alpha\beta\ (\ cosh\ (\ \alpha\ )\ cos\ (\ \alpha\ )\ cos\ (\ \alpha z\ )\ sinh\ (\ \alpha z\ )$$

$$+\ sin\ (\ \alpha\ )\ cosh\ (\ \alpha\ )\ sin\ (\ \alpha z\ )\ sinh\ (\ \alpha z\ )$$

$$+\ cosh\ (\ \alpha z\ )\ sin\ (\ \alpha z\ )\ cos\ (\ \alpha\ )\ cosh\ (\ \alpha\ )$$

$$-\ sinh\ (\ \alpha z\ )\ sinh\ (\ \alpha\ )\ stn\ (\ \alpha\ )\ cos\ (\ \alpha z\ )\ )$$

$$-\ 2\alpha^2\beta^2\ (\ cosh\ (\ \alpha z\ )\ cos\ (\ \alpha z\ )\ cos\ (\ \alpha\ )\ cosh\ (\ \alpha\ )$$

$$+\ sinh\ (\ \alpha z\ )\ sin\ (\ \alpha z\ )\ sin\ (\ \alpha\ )\ sinh\ (\ \alpha\ )\ )$$

$$-\ 2\alpha^3\beta^3\ (\ cosh\ (\ \alpha\ )\ cos\ (\ \alpha\ )\ cos\ (\ \alpha z\ )\ sinh\ (\ \alpha z\ )$$

$$+\ sin\ (\ \alpha\ )\ sin\ (\ \alpha z\ )\ sinh\ (\ \alpha z\ )\ cosh\ (\ \alpha\ )$$

$$-\ cos\ (\ \alpha\ )\ cosh\ (\ \alpha\ )\ cosh\ (\ \alpha z\ )\ sin\ (\ \alpha z\ )$$

$$+\ sin\ (\ \alpha\ )\ sinh\ (\ \alpha\ )\ sinh\ (\ \alpha z\ )\ cos\ (\ \alpha z\ )\ )$$

$$+\ (\ sinh\ (\ \alpha\ )\ sin\ (\ \alpha\ )\ cosh\ (\ \alpha z\ )\ cos\ (\ \alpha z\ )$$

$$-\ cos\ (\ \alpha\ )\ cosh\ (\ \alpha\ )\ sin\ (\ \alpha z\ )\ sinh\ (\ \alpha z\ )\ )\ ]\ /\ [\ \alpha^2\ (\ 1\ +\ 4\alpha^4\beta^4\ )$$

$$(\ sinh^2\ \alpha\ +\ cos^2\ \alpha\ )\ ] \qquad (10)$$

$$\frac{u_y}{P_0} = \Big\{ [\ \alpha\beta\ (\ cosh\ (\ \alpha\ )\ cos\ (\ \alpha\ )\ cos\ (\ \alpha z\ )\ sinh\ (\ \alpha z\ )$$

$$+\ sin\ (\ \alpha z\ )\ sinh\ (\ \alpha z\ )\ sin\ (\ \alpha\ )\ cosh\ (\ \alpha\ )$$

$$-\ cos\ (\ \alpha\ )\ cosh\ (\ \alpha\ )\ sin\ (\ \alpha z\ )\ cosh\ (\ \alpha z\ )$$

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$$+ \sin(\alpha)\sinh(\alpha)\sinh(\alpha z)\cos(\alpha z))$$

$$- 2\alpha^2\beta^2(\sinh(\alpha z)\sin(\alpha z)\cos(\alpha)\cosh(\alpha)$$

$$- \sin(\alpha)\sinh(\alpha)\cos(\alpha z)\cosh(\alpha z))$$

$$- 2\alpha^3\beta^3(\cos(\alpha)\cosh(\alpha)\cos(\alpha z)\sinh(\alpha z)$$

$$+ \sin(\alpha)\sin(\alpha z)\sinh(\alpha z)\cosh(\alpha)$$

$$+ \sin(\alpha z)\cosh(\alpha z)\cos(\alpha)\cosh(\alpha)$$

$$- \sin(\alpha)\sinh(\alpha)\sinh(\alpha z)\cos(\alpha z))$$

$$+ (\cos(\alpha)\cosh(\alpha)\cos(\alpha z)\cosh(\alpha z)$$

$$+ \sin(\alpha)\sinh(\alpha)\sin(\alpha z)\sinh(\alpha z))]$$

$$/[\alpha^2(1+4\alpha^4\beta^4)(\sinh^2\alpha+\cos^2\alpha)]\Big\} - \frac{1}{a^2} \qquad (11)$$

where

$$P_0 = \tfrac{1}{2}\alpha^2 P \qquad (12)$$

Now, for small $\alpha$, we expand in powers of $\alpha$,

$$\frac{u_x}{P_0} = (1 - 2\beta^2 - 2\beta z - z^2) + \alpha^4[(8\beta^6 - 8\beta^4 - 4\beta^3$$

$$- \frac{2}{3}\beta^2 - \frac{7}{3}\beta - \frac{61}{90}) + (8\beta^5 - 4\beta^3 - \frac{4}{3}\beta)z$$

$$- (-4\beta^4 + 2\beta^2 - \frac{5}{6})z^2 - \frac{2}{3}\beta(1-2\beta^2)z^3$$

$$- \frac{(1-2\beta^2)}{6}z^4 + \beta\frac{z^5}{15} + \frac{z^6}{90}] + \alpha^8[(-32\beta^{10}$$

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$$+ 32\beta^8 + 16\beta^7 - \frac{16}{3}\beta^6 + \frac{4}{3}\beta^5 + \frac{28}{3}\beta^4 + \frac{10}{3}\beta^3$$

$$- \frac{31405}{180\times 73}\beta^2 - \frac{\beta}{90} + \frac{50521}{1800\times 63}) + (-32\beta^9 + 32\beta^7$$

$$+ 16\beta^6 - \frac{16}{3}\beta^5 + \frac{4}{3}\beta^4 + \frac{18}{3}\beta^3 - \frac{17441}{180x63}\beta)z$$

$$+ (16\beta^6 - 16\beta^8 + 8\beta^5 - \frac{8}{3}\beta^4 + \frac{2}{3}\beta^3 + \frac{9}{3}\beta^2 - \frac{16}{45}\beta$$

$$- \frac{1385}{360\times 7})z^2 - \frac{1}{3}(16\beta^7 - 16\beta^5 - 8\beta^4 - \frac{4}{3}\beta^3$$

$$- \frac{14}{3}\beta^2 - \frac{7}{3}\beta)z^3 - \frac{1}{6}(8\beta^6 - 8\beta^4 - 4\beta^3 - \frac{2}{3}\beta^2$$

$$- \frac{7}{3}\beta - \frac{61}{90})z^4 - \frac{1}{30}(8\beta^5 - 4\beta^3 - \frac{4}{3}\beta)z^5$$

$$+ \frac{1}{90}(-4\beta^4 + 2\beta^2 - \frac{5}{6})z^6 + \frac{\beta(1-2\beta^2)}{45\times 7}z^7$$

$$+ \frac{(1-2\beta^2)}{360\times 7}z^8 - \frac{\beta z^9}{180\times 63} - \frac{z^{10}}{1800\times 63}] + \cdots \qquad (13)$$

$$\frac{u_v}{P_0} = \alpha^2[(2\beta^2 - 4\beta^4 - \frac{5}{6}) = 2\beta(1-2\beta^2)z$$

$$+ (1-2\beta^2)z^2 - \frac{2\beta}{3}z^3 - \frac{1}{6}z^4] + \alpha^6[(16\beta^8$$

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$$-16\beta^6 - 8\beta^5 + \frac{8}{3}\beta^4 - \frac{2}{3}\beta^3 - \frac{9}{3}\beta^2 + \frac{16}{45}\beta$$

$$+\frac{1385}{360\times 7}) + (16\beta^7 - 16\beta^5 - 8\beta^4 - \frac{4}{3}\beta^3 - \frac{14}{3}\beta^2$$

$$-\frac{7}{3}\beta)z + (8\beta^6 - 8\beta^4 - 4\beta^3 - \frac{2}{3}\beta^2 - \frac{7}{3}\beta - \frac{61}{90})z^2$$

$$+\frac{1}{3}(8\beta^5 - 4\beta^3 - \frac{4}{3}\beta)z^3 - \frac{1}{6}(2\beta^2 - 4\beta^4 - \frac{5}{6})z^4$$

$$-\frac{\beta(1-2\beta^2)}{15}z^5 - \frac{(1-2\beta^2)}{90}z^6 + \frac{\beta z^7}{45\times 7} + \frac{z^8}{360\times 7}] + \ldots (14)$$

In the absence of rotation, the problem reduces to the flow through a straight channel under a constant pressure gradient in slip flow regime, *i e.*,

$$u_x = P_0(1-2\beta^2-2\beta z-z^2),\ u_y = 0 \qquad (15)$$

The flux $Q_x$ per unit breadth in the direction of the $x$—axis is given, for small $\alpha$, by

$$Q_x = \int_{-1}^{1} u_x\, dz$$

$$= 2P_0(\frac{2}{3} - \beta - 2\beta^2) + 2P_0\alpha^4(8\beta^6 + 4\beta^5$$

$$-\frac{20}{3}\beta^4 - \frac{17}{3}\beta^3 - \frac{19}{15}\beta^2 - \frac{142}{45}\beta - \frac{136}{315})$$

$$+ 2P_0\alpha^8(-32\beta^{10} - 16\beta^9 + \frac{80}{3}\beta^8 + \frac{92}{3}\beta^7$$

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$$+\frac{116}{15}\beta^6+\frac{118}{45}\beta^5+\frac{1054}{105}\beta^4+\frac{1663}{252}\beta^3$$

$$-\frac{3716}{2835}\beta^2-\frac{69831}{1800\times 63}\beta+\frac{44224}{155925})+\cdots \qquad (16)$$

The drag $D_x$ in the direction of the $x$—axis per unit area on the boundary $z=1$ is found to be

$$2P_0(1+\beta)+2P_0\alpha^4(-4\beta^5-4\beta^4+\frac{4}{3}\beta^2+\frac{3}{2}\beta-\frac{8}{15})$$

$$+2P_0\alpha^8(16\beta^9+16\beta^8-8\beta^7-\frac{64}{3}\beta^6-\frac{38}{3}\beta^5$$

$$-\frac{68}{15}\beta^4-\frac{269}{45}\beta^3-\frac{354}{63}\beta^2-\frac{10879}{11340}\beta+\frac{992}{2835})+\cdots \qquad (17)$$

Similarly, the drag $D_x$ at $z=-1$ is

$$2P^0(1-\beta)+2P_0\alpha^4(4\beta^5-4\beta^4+\frac{4}{3}\beta^2-\frac{3}{2}\beta-\frac{8}{15})$$

$$+2P_0\alpha^8(-16\beta^9+16\beta^8+8\beta^7-\frac{16}{3}\beta^6-\frac{10}{3}\beta^5$$

$$+\frac{72}{16}\beta^4+\frac{89}{45}\beta^3-\frac{10}{9}\beta^2+\frac{1303}{11340}\beta+\frac{992}{2835})+\cdots \qquad (18)$$

The resistance coefficient becomes approximately

$$\frac{\gamma_r}{\gamma}=(\frac{Q}{Q_\infty})^2$$

$$=1-2\alpha^4A+\alpha^8(2A^2-2B)+\cdots \qquad (19)$$

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where

$$A = (8\beta^6 + 4\beta^5 - \frac{20}{3}\beta^4 - \frac{17}{3}\beta^3 - \frac{19}{15}\beta^2 - \frac{142}{45}\beta - \frac{136}{315}) / (\frac{2}{3} - \beta - 2\beta^2),$$

$$B = (-32\beta^{19} - 16\beta^9 + \frac{80}{3}\beta^8 + \frac{92}{3}\beta^7 + \frac{116}{15}\beta^6 + \frac{118}{45}\beta^5 + \frac{1054}{105}\beta^4 + \frac{1663}{252}\beta^3 - \frac{3716}{2835}\beta^2 - \frac{69831}{18006\times 3}\beta + \frac{44224}{155925}) / (\frac{2}{3} - \beta - 2\beta^2) \quad (20)$$

$\gamma$ and $\gamma_r$ being the resistance coefficients for the flows without and with rotation, respectively, and $Q$ and $Q_x$ being the flux in these cases.

When $\mathfrak{R}$ is large, from (10) and (11), we obtain, for $1 \geqslant z > 0$,

$$u_x \simeq \frac{P_0 e^{\alpha(z-1)}}{\alpha^2(1+4\alpha^4\beta^4)} [(1+\beta\alpha - 2\alpha^3\beta^3) \sin\alpha(1-z) - (\alpha\beta - 2\alpha^2\beta^2 - 2\alpha^3\beta^3)\cos\alpha(1-z)] \quad (21)$$

$$u_y \simeq \frac{P_0}{\alpha^2}[-1 + \frac{e^{\alpha(z-1)}}{(1+4\alpha^4\beta^4)}\{(\alpha\beta + 2\alpha^2\beta^2) + 2\alpha^3\beta^3)\sin\alpha(1-z) + (1+\alpha\beta - 2\alpha^3\beta^3)\cos\alpha(1-z)\}] . \quad (22)$$

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Similar expressions may be written for $0 > z \geqslant -1$.

## 3. DISCUSSION AND CONCLUSIONS

The graphs of $u_x$ and $u_y$ are plotted in Figures 1 and 2. Figure 1 shows that, as $\beta$ increases, the primary flow $u_x$ decreases. From Figure 2, it is inferred that the secondary flow $u_y$ also decreases in magnitude with an increase in $\beta$.

When there is no rotation, the flux in the $x$-direction and the resistance coefficient are found to be independent of slip parameter, However, in the presence of rotation, as $\beta$ increases (i) the flux decreases, and (ii) the resistance coefficient increases. For fixed $\beta$, as $\alpha$ increases (i) the flux $Q_x$ decreases, and (ii) the resistance coefficient increases. For fixed rotation, as $\beta$ increases the drag at the upper wall increases, while at the lower wall it decreases. As $\alpha$ increases, the drag at both the walls decreases. It should be pointed out that these results hold true for small values of rotation only.

## 4. SUGGESTIONS FOR EXPERIMENTAL VERIFICATIONS

It may be possible to perform experiments by rotating a channel of finite width $B$ which is great compared with the depth $2L$. In such a channel, the conditions close to the walls $z' = \pm L$ are not given by the above calculations, but if the side walls are in such a direction that there is no total flow across them, then the conditions can be attained approximately over most of the channel. It is necessary to keep the side walls at an angle $-\Psi$ with the $x$-axis.where

$$\tan \Psi = \frac{-\int_{-1}^{1} u_y \, dz}{\int_{-1}^{1} u_x \, dz}$$

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$$= [ - ( \beta - 2\alpha^2\beta^3 ) ( \sin ( \alpha ) \cosh ( \alpha ) - \sinh ( \alpha ) \cos ( \alpha ) )$$

$$\sin ( \alpha ) \cosh ( \alpha ) + 2\alpha\beta^2 \sin ( \alpha ) \cos ( \alpha )$$

$$- \sinh ( \alpha ) \cosh ( \alpha ) ) - 1/\alpha ( \sin ( \alpha ) \cos ( \alpha )$$

$$+ \sinh ( \alpha ) \cosh ( \alpha ) ) + 2 ( 1 + 4\alpha^4\beta^4 )$$

$$( \sinh^2 \alpha + \cos^2 \alpha ) ) ] /$$

$$[ - ( \beta + 2\alpha^2\beta^3 ( \sin ( \alpha ) \cosh ( \alpha ) - \text{stnh} ( \alpha ) \cos ( \alpha )$$

$$\sin ( \alpha ) \cosh ( \alpha ) ) - 2\alpha\beta^2 ( \sin ( \alpha ) \cos ( \alpha )$$

$$+ \sinh ( \alpha ) \cosh ( \alpha ) ) + 1/\alpha ( \sinh ( \alpha ) \cosh ( \alpha )$$

$$- \sin ( \alpha ) \cos ( \alpha ) ) ] . \qquad (23)$$

Figure 3 gives the angle $\beta$ versus $\alpha$ for $\beta = 0$ and 0.05. From this graph, we conclude that the angle $\Psi$ for any $\alpha$ increases as $\beta$ increases. Thus the effect of the slip parameter does not inhibit the secondary flow through the side walls.

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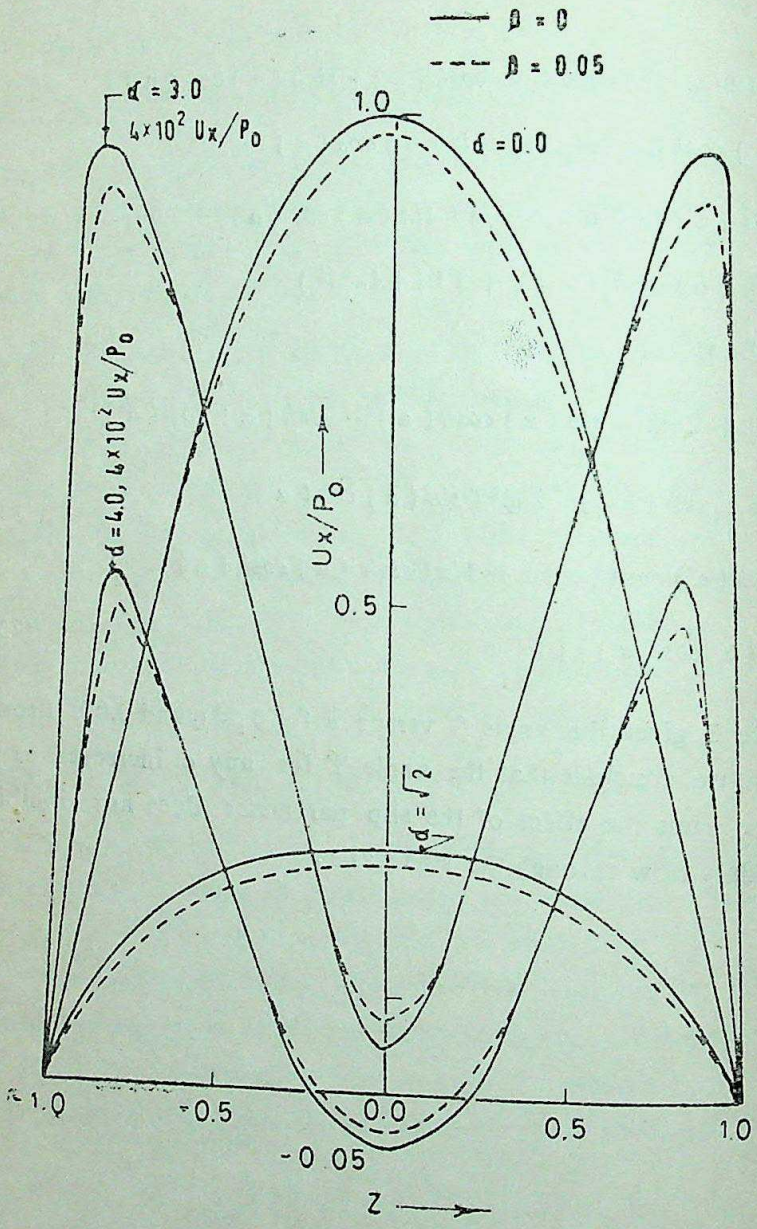


FIG 1 VELOCITY IN THE DIRECTION OF PRESSURE GRADIENT

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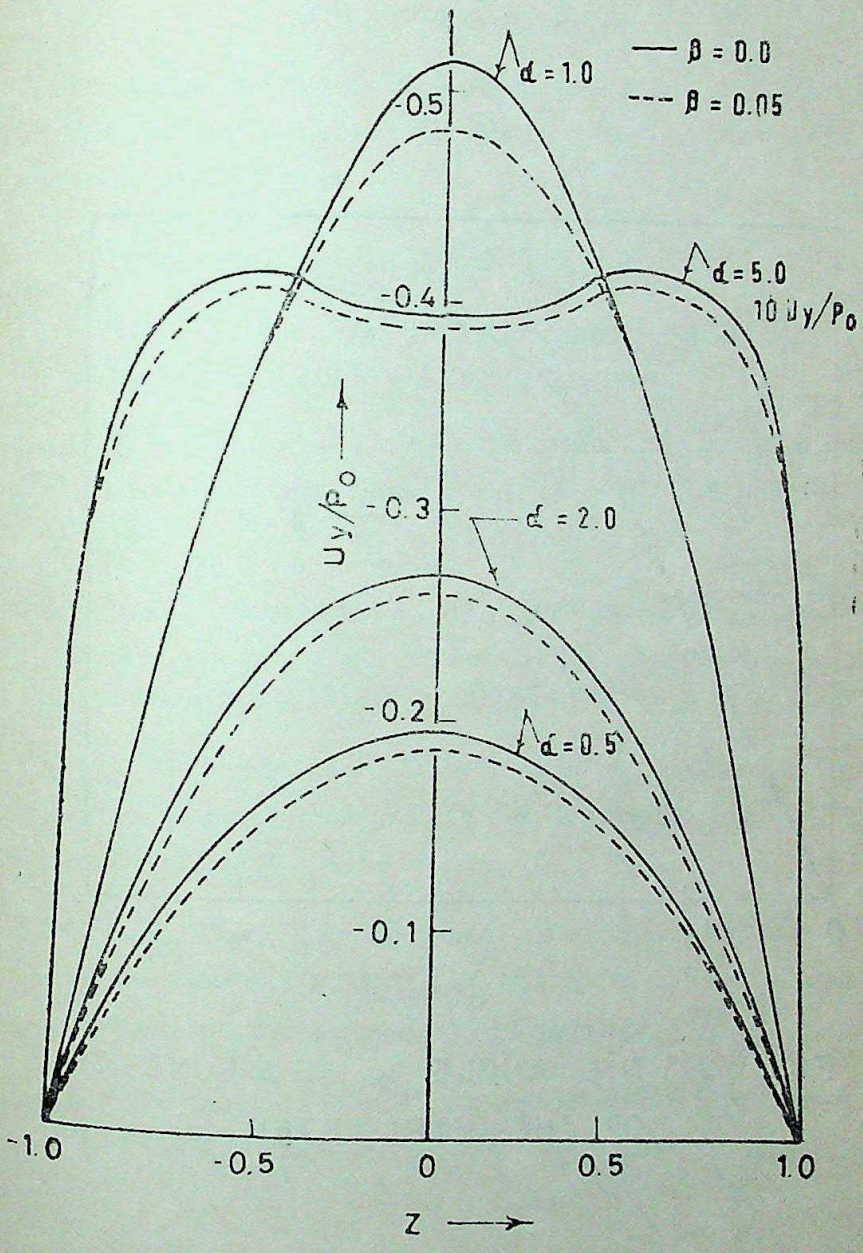


FIG. 2 VELOCITY OF SECONDARY FLOW

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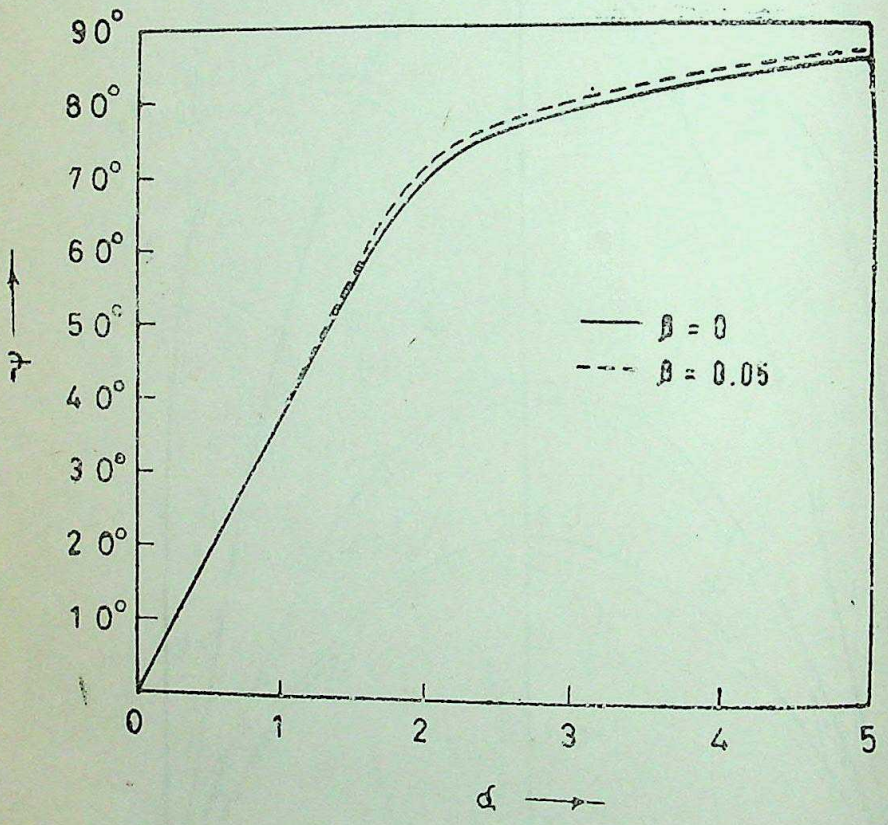


FIG. 3 THE ANGLE $\psi$ AS A FUNCTION OF THE PARAMETER $\alpha$

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# ON A CLASS OF ANALYTIC FUNCTIONS DEFINED BY USING HADAMARD PRODUCT*

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## ABSTRACT

We introduce a class, namely $R^{\delta}_{\lambda,\alpha}(A, B)$ of functions defined by using Hadamard product $f * S_{\sigma}(z)$ of $f(z)$ and $S_{\alpha}(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$. The aim of the present paper is to determine sharp coefficient estimates and maximization theorem concerning the coefficients. Further, we give a sufficient condition in terms of coefficients for functions belonging to the class $R^{\delta}_{\lambda,\alpha}(A, B)$.

## 1. INTRODUCTION

Let A be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $U = z : |z| < 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $q(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belong to $A$, the convolution or Hadamard product of $f(z)$ and $g(z)$ is defined by the power series

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$$f * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \tag{1.1}$$

Let $S$ denote the subclass of A consisting of analytic and univalent functions $f(z)$ in the unit disc U. A function $f(z)$ of $S$ is said to be starlike of order $\alpha$, $0 \leqslant \alpha < 1$, *if*

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \alpha, \; z \in U. \tag{1.2}$$

We denote by $S^*(\alpha)$ the class of all starlike functions of order $\alpha$.

Now, the fnnction

$$S_\alpha(z) = \frac{z}{(1-z)^{2(1-\alpha}} \tag{1.3}$$

is the well known extremai function for the class $S^*(\alpha)$. $S_\alpha(z)$ can be written in the form

$$S_\alpha(z) = z + \sum_{n=2}^{\infty} c(\alpha, n) z^n \tag{1.4}$$

where $c(\alpha, n) = \dfrac{\prod_{k=2}^{n} (k-2\alpha)}{(n-1)!}$, $n = 2, 3, \ldots$ (1.5)

Then we note that $c(\alpha, n)$ is decreasing in $\alpha$ and satisfies

$$\lim_{n\to\infty} c(\alpha, n) = \begin{cases} \infty, & \text{when } \alpha < \frac{1}{2} \\ 1, & \text{when } \alpha = \frac{1}{2} \\ 0, & \text{when } \alpha > \frac{1}{2} \end{cases} . \tag{1.6}$$

In the present note, we introduce a class $R^{\delta}_{\lambda,\alpha}(A.B)$ f functions $f(z)$ belonging to A and satisfying the condition

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$$\left| \frac{e^{i\delta} (f * S_\alpha (\alpha))' - 1}{\lambda (A-B) \cos \delta - Be^{i\delta} (f * S_\alpha (z))' - 1} \right| < 1, z \in U, \tag{1.7}$$

where $A, B, \lambda, \delta$ are arbitrary numbers such that $-1 \leqslant B < A \leqslant 1$, $0 < \lambda \leqslant 1$, $\delta \in (-\pi/2, \pi/2)$. It is easy to see that the condition (1.7) is equivalent to

$$e^{i\delta} . \{f * S_\alpha (z)\}' = (1-\lambda) e^{i\delta} + \cos \delta . \left[ \frac{1+A\omega(z)}{1+B\omega(z)} \right] + i \lambda \sin \delta, \tag{1.8}$$

where $\omega(z)$ is analytic in $U$ and satisfies $\omega(0) = 0$, $|\omega(z)| < 1$.

The main aim of the present paper is to study coefficient estimates, maximization of $|a_3 - \mu a_2^2|$ over the class $R^{\delta}_{\lambda,\alpha}(A,B)$ and a sufficient condition in terms of coefficients for a function belonging to $R^{\delta}_{\lambda,\alpha}(A,B)$. All the results are sharp, Our results generalize the corresponding results of Caplinger and Causey [1], Juneja and Mogra [2], Padmanabhan [5], and Shukla and Dashrath [6].

## 2. LEMMA

The following lemma is due to Keogh and Merkes [3]

*Lemma* 2.1. Let $\omega(z) = \sum_{R=1}^{\infty} c_k z^k$ be analytic in $U$ such that $\omega(0)=0$ and $|\omega(z)| < 1$. If $d$ is any complex number, then

$$|c_2 - d c_1^2| \leqslant \max [1, |d|]. \tag{2.1}$$

Equality may be attained with the functions $\omega(z) = z^2$ and $w(z) = z^2$.

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## 3. COEFICIENT ESTIMATES

Theorem 3.1. If $f(z)$ is in $R^{\delta}_{\lambda,\alpha}(A,B)$, then

$$|a_j| \leqslant \frac{(A-B)\cos\delta}{j.\,c(\alpha,j)}, \quad j=2,3,4,\ldots, \tag{3.1}$$

where $c(\alpha,j) = \dfrac{\prod_{k=2}^{j}(k-2\alpha)}{(j-1)!}$.

The estimate (3.1) is sharp.

**Proof**. Since $f(z) \in R^{\delta}_{\lambda,\alpha}(A,B)$, we have form (1.8) that

$$e^{i\delta}.(f^{*}S_{\alpha}(z))' = (1-\lambda)e^{i\delta} + \cos\delta.\left[\frac{1+A\omega(z)}{1+B\omega(z)}\right] + i\lambda\sin\delta \tag{3.2}$$

where $\omega(z)$ is analytic in $U$ and satisfies the conditions $\omega(0)=0$, $|\omega(z)|<1$ for $z \in U$. From (3.2), we have

$$e^{i\delta}.(f^{*}S_{\alpha}(z))' - 1 = [(A-B)\lambda\cos\delta - Be^{i\delta}(f^{*}S_{\alpha}(z))' - 1]\,\omega(z)$$

or

$$e^{i\delta}.\sum_{n=2}^{\infty} n.c(\alpha,n)\,a_n z^{n-1} = [(A-B)\lambda\cos\delta - Be^{i\delta}.\sum_{n=2}^{\infty} n.c(\alpha,n)\,a_n z^{n-1}]\,\omega(z), \tag{3.3}$$

where $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$,

Equating corresponding coefficients on both sides of (3.3), we find that the coefficient $a_j$ on the left hand side of (3.3) depends on $a_2, a_3, \ldots, a_{j-1}$ on tne right hand side of (3.3) Hence, for $j \geqslant 2$, it follows from (3.3) that

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$$e^{i\delta} \cdot \sum_{n=2}^{j} n.c(\alpha,n)\, a_n z^{n-1} + \sum_{n=j+1}^{\infty} a_n z^{n-1}$$

$$= [\lambda (A-B) \cos\delta - Be^{i\delta} \sum_{n=2}^{J-1} n.c(\alpha,n)\, a_n z^{n-1}]\, \omega(z)$$

where $a_n$ are some complex numbers .

Since $|\omega(z)| < 1$, by using parseval's identity,

$$\sum_{n=2}^{j} n^2 . (c(\alpha,n))^2 . |a_n|^2 . r^{2(n-1)} + \sum_{n=j+1}^{\infty} |a_n|^2 . r^{2(n-1)}$$

$$\leqslant \lambda^2 (A-B)^2 \cos^2\delta + B^2 . \sum_{n=2}^{j-1} n^2 . (c(\alpha,n))^2 . |a_n|^2 . r^2 (n-1)$$

$$\leqslant \lambda^2 (A-B)^2 \cos^2\delta + B^2 . \sum_{n=2}^{j-1} n^2 (c(\alpha,n))^2 . |a_n|^2 . |a_n|^2 .$$

Letting $r \to 1$ on the left hand side of the above inequality, we obtain

$$\sum_{n=2}^{j} n^2 . (c(\alpha,n))^2 . |a_n|^2$$

$$\leqslant \lambda^2 (A-B)^2 \cos^2\delta + B^2 . \sum_{n=2}^{j-1} n^2 . (c(\alpha,n))^2 |a_n|^2 .$$

Thus

$$j^2 . (c(\alpha,j))^2 . |a_j|^2 \leqslant \lambda^2 (A-B)^2 \cos^2\delta -$$

$$-(1-B^2) \sum_{n=2}^{j-1} n^2 (c(\alpha,n))^2 |a_n|^2 \leqslant \lambda^2 (A-B)^2 \cos^2\delta .$$

Hence

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$$| a_j | \leqslant \frac{(A-B)\cos\delta}{j.\, c(\alpha, j)}, \quad j=2,3, \ldots,$$

where $c(\alpha, j) = \dfrac{\prod_{k=2}^{j} (k-2\alpha)}{(j-1)!}$.

To show the sharpness of the result, we eonsider the function $f(z)$ defined by

$$(f * S_\alpha(z))' = 1 + \frac{\lambda(A-B)\cos\delta\, e^{-i\delta} \cdot z^{n-1}}{1+Bz^{n-1}}. \tag{3.1}$$

It is now easy to see that the equality is attained in (3.1).

THEOREM 3.2. If $f(z) \in R^{\delta}_{\lambda,\alpha}(A, B)$.

Then

$$(1-B)^2 \sum_{n=2}^{\infty} n^2 . (c(\alpha, n))^2 . |a_n|^2 \leqslant \lambda^2 (A-B)^2 \cos^2\delta \tag{3.5}$$

**Proof**. Since $f(z) \in R^{\delta}_{\lambda,\alpha}(A,B)$, Then we have form (1.8) that

$$e^{i\delta} . (f * S_\alpha(z))' = (1-\lambda) e^{i\delta} + \cos\delta . \left[\frac{1+A\omega(z)}{1+B\omega(z)}\right] + i\lambda \sin\delta$$

where $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ is analytic in $U$ and satisfies

$\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$. Hence

$$e^{i\delta} . \sum_{n=2}^{\infty} n.c(\alpha,n)\, a_n z^{n-1} = [\lambda(A-B)\cos\delta - Be^{i\delta} . \sum_{n=2}^{\infty} n.c(\alpha,n)\, a_n z^{n-1}]$$

$\omega(z)$,

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On solving, we get

$$\sum_{n=2}^{\infty} n^2 . ( c ( \alpha,n ) )^2 . | a_n |^2$$

$$\leqslant \lambda^2 (A-B)^2 \cos^2 \delta + B^2 \sum_{n=2}^{\infty} n^2 ( c ( \alpha,n ) )^2 | a_n |^2$$

$$( 1-B^2 ) \sum_{n=2}^{\infty} n^2 . ( c ( \alpha,n ) )^2 . | a_n |^2 \leqslant \lambda^2 ( A-B)^2 \cos^2 \delta .$$

THEOREM 3.3. Let $f(z)$ be analytic in $U$. If for $-1 \leqslant B \leqslant 0$,

$$\sum_{n=2}^{\infty} (1-B) . ( c ( \alpha,n ) . n | a_n | \leqslant \lambda (A-B) \cos \delta, \qquad (3.6)$$

then $f(z) \in R^{\delta}_{\lambda,\alpha} (A,B)$.

The result is sharp .

Proof . Suppose that (3.6) holds. Then, for $z \in U$, we have

$$| e^{i\delta} ( f * S_{\alpha}(z) )' - 1 | - | \lambda ( A-B ) \cos \delta - Be^{i\delta} ( f(z) ) - 1 |$$

$$= | e^{i\delta} \sum_{n=2}^{\infty} c ( \alpha,n ) n a_n z^{n-1} | - | \lambda(A-B) \cos \delta - Be^{i\delta} . \sum_{n=2}^{\infty} c (\alpha,n) . n a_n z^{n-1} |$$

$$\leqslant \sum_{n=2}^{\infty} c (\alpha,n) n | a_n | r^{n-1} - \lambda (A-B) \cos \delta - B \sum_{n=2}^{\infty} c (\alpha,n) . n | a_n | r^{n-1}$$

$$< \sum_{n=2}^{\infty} c ( \alpha,n ) ) . n | a_n | - \lambda (A-B) \cos \delta - B \sum_{n=2}^{\infty} c (\alpha,n) . n | a_n |$$

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$$= \sum_{n=2}^{\infty} (1-B)\, c(\alpha,n)\, .\, n\, |\, a_n\, | - \lambda (A-B) \cos\delta \leqslant 0. \text{ by } (3.6).$$

Hence it follows that

$$\left| \frac{e^{i\delta} . (f^* S_\alpha(z))' - 1}{(A-B)\cos\delta - Be^{i\delta} . (f^* S_\alpha(z))' - 1} \right| < 1, z \in U .$$

Therefore $f(z) \in R^{\delta}_{\lambda,\alpha}(A,B)$ .

The function

$$f(z) = z - \frac{\lambda (A-B) \cos\delta . e^{-i\delta}}{(1-B) . c(\alpha,n) . n} z^n, \; n=2, 3, \ldots$$

is an extremal function for this theorem since for this function

$$\left| \frac{e^{i\delta} (f * S_\alpha(\alpha))' - 1}{\lambda (A-B)\cos\delta - Be^{i\delta} (f * S_\alpha(z))' - 1} \right| < 1,$$

## 4. MAXIMIZATION OF $| a_3 - \mu a_2^2 |$

Theorem 4.1. If $f(z) = z + \sum_{n=2}^{\infty} \in R^{\delta}_{\lambda,\alpha}(A,B)$, then, for any complex $\mu$, we have

$$| a_3 - \mu a_2^2 | \leqslant \frac{\lambda (A-B)\cos\delta}{3c(\alpha,3)} \max \{ 1, | d | \}, \qquad (4.1)$$

where

$$d = \frac{4B \sec\delta . e^{i\delta} . (c(\alpha,2))^2 + 3\mu\lambda (A-B) . c(\alpha,3)}{4 \sec\delta . e^{i\delta} . (c(\alpha,2))^2} .$$

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The result is sharp .

**Proof .** Since $f(z) \in R^{\delta}_{\lambda,\alpha}(A,B)$, we have

$$e^{i\delta} . (f * S_{\alpha}(z))' = (1-\lambda) e^{i\delta} + \lambda \cos\delta . \left[\frac{1+A\omega(z)}{1+B\omega(z)}\right] + i\lambda \sin\delta \qquad (4\cdot2)$$

where $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ and satisfies the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in U$. From (4.2), we get

$$\omega(z) = \frac{e^{i\delta} . (f * S_{\alpha}(z))' - 1}{\lambda(A-B)\cos\delta - Be^{i\delta}(f * S(z))' - 1}$$

$$= \frac{\sec\delta\, e^{i\delta}}{\lambda(A-B)} \left[ 2c(\alpha,2)\, a_2 z + 3c)\, a_3 z^2 + \frac{Be^{i\delta}\, 4\,(c(\alpha,2))^2 a_2^2 z^2}{\lambda(A-B)\cos\delta} \right] .$$

Now comparing the coefficients of $z$ and $z^2$ on both sides, we have

$$c_1 = \frac{\sec\delta . e^{i\delta}}{\lambda(A-B)}\, 2c(\alpha,2)\, a_2$$

and $$c_2 = \frac{\sec\delta . e^{i\delta}}{\lambda(A-B)} \left[ 3c(\alpha,3)\, a_3 + \frac{4Be^{i\delta}(c(\alpha,2))^2 a_2^2}{\lambda(A-B)\cos\delta} \right] .$$

Thus $$a_2 = \frac{c_1 \lambda(A-B)}{2\sec\delta . e^{i\delta} . c(\alpha,2)}$$

and $$a_3 = \frac{c_2\lambda(B-B) - Bc_1^2}{3\sec\delta . e^{i\delta} . c(\alpha,3)} .$$

Hence

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$$a_3 - \mu a_2^2$$

$$= \frac{c_2\lambda\,(A-B) - Bc_1^2\,\lambda\,(A-B)}{3sec\,\delta\,.\,e^{i\delta}\,.\,c\,(\alpha,3)} - \mu.\frac{c_1^2\,\lambda^2\,(A-B)^2}{4sec^2\,\delta\,.\,e^{i\delta}.\,e^{2i\delta}\,(\,c\,(\alpha,2\,)\,)^2}$$

$$= \frac{\lambda\,(A-B)\,cos\delta}{3e^{i\delta}\,c\,(\alpha,3)}\,[\,c_2 - c_1^2\,\frac{4Bsec\delta\;e^{i\delta}.\,(c\,(\alpha,2)\,)^2 + 3\mu\lambda\,(A-B)\,c\,(\alpha,3)}{4\,sec\,\delta\,e^{i\delta}\,.\,(c\,(\alpha,2)\,)^2}\,]\,.$$

Therefore

$$|\,a_3 - \mu a_2^2\,| = \frac{\lambda\,(A-B)\,cos\,\delta\,,}{3.\,c\,(\alpha,\,3)}\;|\,c_2 - dc_1^2\,|\,,$$

where

$$d = \frac{4Bsec\delta\,.\,e^{i}\,.\,(\,c\,(\alpha.2\,)\,)^2 + 3\mu\lambda\,(A-B).\,c\,(\alpha,3\,)}{4sec\delta\,.\,e^{i\delta}\,.\,(\,c\,(\alpha,2\,)\,)^2}\,.$$

Applying *Lemma* 2.1. In the above equation, we get,

$$|\,a_3 - \mu a_2^2\,| \leqslant \frac{\lambda\,(A-B)\,cos\,\delta}{3\,c\,(\alpha,3\,)}\;max\,\{1,\,|\,d\,|\,\}\,.$$

Since the inequality (2.1) is sharp, so that the result (4.1) is sharp.

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# A NOTE ON CERTAIN GENERATING RELATIONS INVOLVING A CLASS OF POLYNOMIALS

*By*

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## ABSTRACT

The authors aim at deriving generating relations of Mehler's type by applying a group-theoretic approach.

## 1. INTRODUCTION

In an attempt to unify classical polynomials, Patil and Thakare [1] considered a class of functions defined by

$$(1.1) \quad P_n^{\alpha}(x; r, s, p, k, \lambda) = x^{-\alpha} e^{px^r} \theta_{x,\lambda}^n [x^{\alpha+sn} e^{-px^r}],$$

where

$$\theta_{x,\lambda} = \lambda x^k + x^{k+1} D, \quad D \equiv \frac{d}{dx}.$$

The work of Patil and Thakare and several others, using the differential operator $\theta_{x,\lambda}$ was motivated by the work of Srivastava and Singhal [3], using the special case, $\theta_{x,0} = x^{k+1} D_x$. The above generalization includes classieal and other polynomials and functions of mathematical physics as its special cases, like Hermite, Gegenbauer and Laguerre polynomials and their generallzations. One can refer to

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Srivastava and Manocha [2, pp. 447-448, Problems 16 and 18] for more general situations in which the operator $\theta_{x,\lambda}$ has already been used.

In the present note, we shall derive Mehler's type formulas for the function (1.1) by group-theortic approach when a bilinear generating relation is known.

## 2. LIE OPERATORS FOR $P_n^{\alpha}(x; r, s, p, k, \lambda)$

Patil and Thakare [1] have given the operational relation

$$(2.1)\quad [\theta_{x,\lambda} + \alpha x^k - prx^{k+r}]\, P_n^{\alpha}(x; r, s, p, k, \lambda) = x^{-s} P_{n+1}^{\alpha-s}(x; r, s, p, k, \lambda).$$

Now put

$$(2.2)\quad \Lambda_{y,x} = yx^s [\theta_{x,\lambda} + \alpha x^k - prx^{k+r}],$$

so that

$$(2.3)\quad \Lambda_{y,x} \{y^n P_n^{\alpha}(x; r, s, p, k, \lambda)\} = y^{n+1} P_{n+1}^{\alpha-s}(x; r, s, p, k, \lambda).$$

Hence, clearly $\Lambda_{y,x}$ forms a raising Lie operator, for the class of functions $P_n^{\alpha}(x; r, s, p, k, \lambda)$.

The extended form of this operator is given by

$$(2.4)\quad e^{\phi\Lambda_{y,x}} f(x; y)$$

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$$= (1-t)^{-(\alpha+\lambda)/k} \exp\left[px^r\left\{1-(1-t)^{-r/k}\right\}\right] f\left\{x(1-t)^{-1/k}; y\right\},$$

where $t = \phi\, y\, k\, x^{k+s}$ .

## 3. MEHLER'S TYPE FORMULAE

Consider the bilinear generating function

(3.1) $G(x, z, \phi)$

$$= \sum_{n=0}^{\infty} a_n \phi^n P_n^{\alpha}(x; r, s, p, k, \lambda) P_n^{\beta}(z; e, f, i, j, \delta).$$

Putting $\phi = \phi t_1 t_2 g$ , we get

(3.2) $G(x, z, \phi t_1 t_2 g)$

$$= \sum_{n=0}^{\infty} a_n \phi^n g^n \left\{ t_1^n P_n^{\alpha}(x; r, s, p, k, \lambda) \right\}$$

$$t_2^n P_n^{\beta}(z; e, f, i, j, \delta)\} .$$

Now operating upon both sides by $\{(\exp \phi \Lambda_{x,t_1})$ $(\exp \phi \Lambda_{z,t_2})\}$ and with appropriate adjustments of parameters, we find, using (2.3), that

(3.3) $(\exp \phi \Lambda_{x,t_1})(\exp \phi \Lambda_{z,t_2})\, G(x, z, \phi t_1 t_2 g)$

$$= \sum_{n=0}^{\infty} a_n \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \frac{\phi^{l_1}}{l_1!} t_1^{l_1+n} P_{n+l_1}^{\alpha-l_1 s}(x; r, s, p, k, \lambda)$$

$$\frac{\phi^{l_2}}{l_2!} t_2^{l_2+n} P_{n+l_2}^{\alpha-l_2 f}(z; e, f, i, j, \delta)(\phi g) .$$

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But, by (2.4),

$$(3.4)\quad (\exp \phi \Lambda_{x,t_1})(\exp \phi \Lambda_{z,t_2})\, G(x, z, \phi t_1 t_2 g)$$

$$= (1 - \phi t_1 k x^{k+s})^{-(\alpha+\lambda)/k} \exp[\, p x^r \{ 1 - (1 - \phi t_1 k x^{s+k})^{-r/k} \}$$

$$(1 - \phi t_2 j z^{f+j})^{-(\beta+\delta)/j} \exp[\, i z^e \{ 1 - (1 - \phi t_2 j z^{f+j})^{-e/j} \}$$

$$G[\, x(1 - \phi t_1 k x^{k+s})^{-1/k},\ z(1 - \phi t_2 j z^{f+j})^{-1/j},\ \phi t_1 t_2 g\,].$$

From (3.3), and (3.4) on putting $t_1 = t_2 = 1$ and after some manipulations, we get

$$(3.5)\quad (1 - \psi k x^{k+s})^{-(\alpha+\lambda)/k} \exp[\, p x^r \{ 1 - (1 - \phi k x^{s+k})^{-r/k} \}$$

$$(1 - \phi j z^{f+j})^{-(\beta+\delta)/j} \exp[\, i z^e \{ 1 - (1 - \phi j z^{f+j})^{-e/j} \}$$

$$G[\, x(1 - \phi k x^{k+s})^{-1/k},\ z(1 - \phi j z^{f+j})^{-1/j},\ \phi g\,]$$

$$= \sum_{n=0}^{\infty} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\min(n,l_1)} \frac{a_{n-l_2}\, \phi^{n+l_1-l_2}\, g^{n-l_2}}{(l_1-l_2)!\ l_2!}$$

$$P_{n+l_1-2l_2}^{a-(l_1-l_2)s}(x; r, s, p, k, \lambda)\, P_n^{a-l_2 f}(z; e, f, i, j, \delta)$$

Thus we have proved the theorem :

If there exists a bilinear generating function of the form (3 1), then there exists a generating relation of the form (3.5).

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## 4. APPLICATION OF THE THEOREM

Although the above theorem can be applied to many classical polynomials, we give below a special case involving Hermite polynomials .

We start from the following bilinear generating function for Hermite polynomials $H_n(xy)$, known as Mehler's formula :

$$(4.1) \quad \sum_{n=0}^{\infty} H_n(x)\, H_n(y)\, \frac{t^n}{2^n n!} = (1-t^2)^{-1/2} \exp\left\{ \frac{2xyt - t^2(x^2+y^2)}{1-t^2} \right\}$$

Hence, from (1.1) (3.1) and (3.5), we get

$$(4.2) \quad (1-t^2g^2)^{-1/2} e^{\{x^2-(x+t)^2\}} e^{\{z^2-(z+t)^2\}}$$

$$\exp\left\{ \frac{2(x+t)(z+t)tg - \{(x+t)^2+(z+t)^2\} t^2g^2}{1-t^2g^2} \right\}$$

$$= \sum_{n=0}^{\infty} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{min(n,l_1)} \frac{(-1)^{l_1}\, t^{n+l_1-l_2}\, g^{n-l_2}}{2^{n-l_2}(n-l_2)!\,(l_1-l_2)!}$$

$$H_{n+l_1-2l_2}(x) \cdot H_n(z) .$$


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# A NOTE ON SCATTERED SPACES

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The purpose of this note is to present some results on scattered spaces. The principal theorems concern scattered spaces and inverse limits. We shall use the following nomenclature and notation. Let $X$ be a topological space and let $A \subset X$. Then $A'$ is the set of accumulation points of $A$. The symbol $\Phi$ denotes the empty set. Also, we shall assume that all spaces are non empty.

**Definiton.** A topological space $X$ is scattered if and only if for any non-empty subset $A \subset X$, $A$ is not contained in $A'$. That is, there is a point $x \in A$ which is not a accumulation point of $A$.

A simple example of a compact, scattered space is the set $X = \{0\} \cup \{1/n : n \geqslant 1\}$ with the relative topology from the reals.

Remark. It is fairly easy to show that a $T_1$-space which is scattered is totally disconnected ( see [3], page 33 ). Further, Example 57 of [3] is a connected scattered space. Finally, this remark and Proposition 8.6, page 46, of [2] gives the following

**Proposition 1.** If $X$ is a compact, scattered, Hausdorff space, then $\dim X \leqslant 0$.

The next example shows that if we drop the assumption that $X$ is compact, then the dimension of $X$ need not be less than or equal to 0 at at every point.

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**Example 1.** Define the following three sets in the plane. Let $X_1 = \{ (0, 1/n) : n \geqslant 1 \}$, $X_2 = \{ (1/m, 0) : m \geqslant 1 \}$ and $X_3 = \{ (1/m, 1/n) : m \geqslant 1, n \geqslant 1 \}$. Then set $X = \{ (0,0) \} \cup X_1 \cup X_2 \cup X_3$. Define basic open sets for $X$ as follows. For $(0, 1/n)$ a basic open set has the form $\{ (0, 1/n) \} \cup \{ (1/m, 1/n) : m \geqslant k \}$ and for $(1/m, 0)$ the basic open sets have the form $\{ (1/m, 0) \} \cup \{ (1/m, 1/n) : n \geqslant k \}$. Next the sets $\{ (1/m, 1/n) \}$ are all open. Finally, a basic open set of $(0,0)$ has the form $\{ (0, 0) \} \cup \{ (1/m, 1/n) : n \geqslant i, m \geqslant j \}$. Then $X$ is $T_2$ but not compact. Moreover, each of the points $(0, 1/n)$ or $(1/m, 0)$ is a limit point of $X_3$. However, $X$ is scattered, but dim $X$ is not less than or equal to 0.

Remark. If $ind\, X \geqslant 1$ at each point of a $T_2$-space $X$, then $\{x\}$ is not open for any $x \in X$. $\in X$. Thus, $x \in X'$ for all $x \in X$ and $X$ is not scattered.

In this note a directed set $(D, \leqslant)$ is a non-empty set $D$ and a partial order on $D$ such that if $a, b \in D$, there is a $c \in D$ with $a \leqslant c$ and $b \leqslant c$. Let $(D, \leqslant)$ be a directed set and let $X_d$ be a topological space for each $d \in D$. Further, suppose that for each pair $a, b \in D$, $a \leqslant b$, there is a function $f_{ab} : X_b \to X_a$, with $f_{aa}$ the identity on $X_a$, for all $a \in D$. We shall also assume that if $a \leqslant c \leqslant b$, then $f_{ab} = f_{ac} \circ f_{cb}$. Then the collection of spaces $X_d$, the directed set $D$ and the set of functions form an inverse limit system. Designate an inverse limit system by IS. The inverse limit of the system is the space $X_\infty$ of all members of the product $X = \Pi \{ x_d : d \in D \}$ which satisfy $f_{ab}(\pi_b(x)) = \pi_a(x)$ for all $a, b \in D$ with $a < b$. The space $X_\infty$ has the relative product topology. In general, the space $X_\infty$ is quite different from the spaces $X_d$. In the sequel we shall use $\pi_a$ to mean the projection restricted to $X_\infty$ when it is applied to $X_\infty$. A discussion of inverse

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limits can be found in [1] . In particular, we have ( Theorem 6. b. 11, page 171 of [1] ) .

**Thcorem 2.** If each of the spaces $X_d$ is compact and $T_2$, then $X_\infty$ ii non-empty .

Further, we have the elementary

**Lemma 3.** If $x, y \in X_\infty$ and if $x_b = y_b$ for some $b \in D$, then $x_a = y_a$ for all $a \in D$ with $a < b$ .

**Proof .** This follows from the fact that

$$x_a = f_{ab}(x_b) = f_{ab}(y_b) = y_a .$$

Further, basic open set in $X_\infty$ have a very simple form. In Lemma 4, the projections are restricted to $X_\infty$ .

**Lemma 4.** Let $B = \{\pi_d^{-1}(U_d) : d \in D, U_d \subset X_d \text{ an open set}\}$ .

**Proof .** A basic open set in the product topology is the intersection of the inverses of an open set in each of a finite number of coordinate spaces . Thus let $d_1, \ldots, d_n$ be a finite subset of $D$ and $U_{d_i}$, $i = 1, \ldots, n$, be open sets. Now let $d \in D$ be such that $d_i \leqslant d$ for all $i = 1, \ldots, n$ . Then if $x \in \cap \{\pi_i^{-1}(U_i) : i = 1, \ldots, n\} = U$, set $U_d = \cap \{f_{d_i}^{-1} d(U_{d_i}) : i = 1, \ldots, n\}$, then $x_d \in U_d$ and $\pi_d^{-1}(U_d) \subset U$ . Thus, the collection B is a basis for the topology on $X_\infty$ ( Also, see Theorem 6 B.B, page 171 of [1] ) .

We shall also ndeed the following

**Lemma 5.** Let $X, Y$ be topological spaces and let $f : X \to Y$ be a continuous function . If $x \in A \subset X$, $f(x) = y$ and if $y$ is not an

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accumulation point of $f(A)$, then there is an open set $U$, $x \in U$, such that $f(A \cap U) = \{y\}$.

**Proof.** Since y is not an accumulation point of $f(A)$, there is an open set $V$, $y \in V$, such that $f(A) \cap V = \{y\}$. Let $U$, $x \in U$, be an open set in $X$ such that $f(U) \subset V$. Then $f(A \cap U) \subset f(A) \cap f(U) \subset f(A) \cap V = \{y\}$.

We shall use the following definition in the first of the principal results.

**Definition.** An inverse limit system IS *splits* at the point $x_a$ in the space $x_a$ if and only if given $b \in D$, $a < b$, $f_{ab}^{-1}(x_a) \neq \Phi$ and if $x_b \in f_{ab}^{-1}(x_a)$, there is a point $c \in D$ with $b < c$ such that $f_{cb}^{-1}(x_b)$ contains at least two points.

We shall need the next lemma for the first main theorem.

**Lemma 6.** Suppose that each of the spaces $x_d$ is compact and $T_2$ and suppose that the system IS splits at the point $x_a$ in $x_a$. Then the set $A(x_b) = \{x \in X_\infty : \pi_b(x) = x_b\} \neq \Phi$ where $d \in D$ with $a \leqslant d$ and $x_b \in f_{ab}^{-1}(x_a)$.

**Proof.** Let $X$ denote the cartesian product of the spaces $X_d$, $d \in D$. Let $b$ be such that $a < b$ and let $x_b \in f_{ab}^{-1}(x_a)$ be fixed. Then for $c$, $b \leqslant c$, set $A_c(x_b) = \{x \in X : \pi_b(x) = x_b$ and $\pi_d(x) = f_{dc}(\pi_c(x))$ for all $d < c\}$. Note that if $x \in f_{be}^{-1}(x_c)$, we can define a point x by

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setting $\pi_c(x) = x_c$ and $\pi_d(x) = f_{dc}(x_c)$ for $d \leqslant c$. If $d \in D$ is not less than or equal to $c$. thne set $\pi_d(x)$ equal to any point in $X_d$. Thus, the sets $A_c(x_b)$ are non-empty. If $x$ is not an element of $A_c(x_b)$, tnen there is a $d \leqslant c$ such that $\pi_d(x) \neq f_{dc}(\pi_c(x))$. Then there are two disjoint open sets $U, V$ in $X_d$ such that $\pi_d(x) \in U$ and $f_{dc}(\pi_c) \in V$. Let $W \subset X$ be $\pi_c^{-1}(f_{de}^{-1}(V)) \cap \pi_d^{-1}(U)$. We see that $x \in W$ and that $W \cap A_c(x_b) = \Phi$. Hence, the sets $A_c(x_b)$ are closed. Moreover, if $b < d < c$, then $A_c(x_b) \subset A_d(x_g)$ and thus, the collection $\{A_c(x_b) : b < c\}$ has the finite intersection property. Therefore the set $A(x_b) = \cap\{A_c(x_b) : b < c\} \neq \Phi$.

Our first result concerning iuverse limits and scattered spaces is the following.

**Theorem 7.** Suppose that each of the spaces $X_d$, $d \in D$ is compact. $T_2$. If the inverse system IS splits at $x_a \in X_a$ for some $a \in D$, then there is a set $A \subset X_\infty$ such that $A \subset A'$.

**Proof.** Let $A = A(x_a)$ from Lemma 6. Let $x \in A$. Set $x_d = \pi_d(x)$ for all $d \in D$. Now let $U$ be a basic open set which contains $x$. Then by Lemma 4, $U = \pi_b^{-1}(U_b)$ where $U_b$ is an open subset of $x_b$ which contains $x_b$. If $b \leqslant a$, then $\pi_b(y) = f_{ba}(\pi_a(y)) = f_{ba}(x_a) = x_b$ for all $y \in A$. Hence, $A \subset U$. So, we may assume that $a \leqslant b$. Now, since the system splits at $x_a$, th e is a $b < c$, such that $f_{bc}^{-1}(x_b)$ contains at least two points. Thus, from the construction in the proof of Lemma 6. we see that there is a point $y \neq x$ in $A$ and $y_c \in f_{bc}^{-1}(x_b)$. Since $y_b = x_b$, $y \in U$. Thus, $x \in A'$.

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Corollary 8. If each $X_d$ is compact. $T_2$ and if the system splits at some $x_a$, then $X_\infty$ is not scattered.

If the directed set has a largest element $\omega$, then many of the properties of $X_\infty$ are determined by the properties of $X\omega$. In particular we have

**Theorem 9.** If $\omega \in D$ is the largest element of $D$ and if $X\omega$ is scattered, then $X_\infty$ is scattered.

**Proof.** Let A be a non-empty subset of $X_\infty$ and let $A_\omega = \pi_\omega(A)$. Since $X_\omega$ is scattered, there is a $x_\omega \in A_\omega$ which is not an element of $A'_\omega$. Let $U_\omega$ be an open set with $x_\omega \in U_\omega$ such that $U_\omega \cap A_\omega = \{x_\omega\}$ and let $x \in A$ with $\pi_\omega(x) = x_\omega$, Then by Lemma 5 there is an open set $V \subset X_\infty$ such that $x \in V \subset U$ and $V \cap A = \{x\}$. Hence. $x$ is not an element of $A'$, and so, $X_\infty$ is scattered.

Next we need the following definition.

**Definition.** The inverse system IS is eventually one-to-one if and only if for every $a \in D$ and $x_a \in X_b$, there is a $b \in D$. $a < b$, such that $f_{cb}^{-1}(x_b)$ contains at most one point for all $x_b \in f_{ab}^{-1}(x_a)$ and for all $c \in D$ with $b < c$.

Now we have a partial converse of Theorem 7.

**Theorem 10** If the system IS is eventually one-to-one and if each of the spaces $X_d$, $d \in D$, is scattered, then $X_\infty$ is scattered.

**Proof.** Let $A$ be a non-empty subset of $X_\infty$, let $a \in D$ and set $A_a = \pi_a(A)$. Since $X_a$ is scattered, there is an $x_a \in A_a - A'_a$. Thus, there is an open set $U_a$ of $X_a$ such that $U_a \cap A_a = (x_a)$. Next there is a $b \in D$, $a < b$, such that for any $x_b \in f_{ab}^{-1}(x_a)$, and any $c$ in $D$,

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$b < c$, $f_{bc}^{-1}(x_b)$ contains at most one point. Set $B = \pi_b(A) \cap f_{ab}^{-1}(x_a)$. Then $\pi_b(x) \in B$ where $x \in \pi_a^{-1}(x_a) \cap A$. Since $X_b$ is scattered, there is a point $x_b \in B$ and an open set $U_b$ of $X_b$ such that $U_b \cap B = \{x_b\}$. Further, $U_b$ can be choosen so that $f_{ab}(U_b) \subset U_b$. Next set $U = \pi_b^{-1}(U_b)$. If $y \in U \cap A$, then $y_b = \pi_b(y) \in B$. Therefore $y_b = x_b$ since $\pi_a(y) = x_a$. If $z \in U \cap A$, then $z_b = x_b$. Further, if $c \in D$, $b < c$, then $f_{cb}^{-1}(x_b)$ contains at most one point and hence, $z_c = y_c$. Moreover, if $d < b$, $z_d = y_d$ by Lemma 3. Finally, if $d$ is any element of $D$, there is an element $b$ in $D$ such that $b < k$ and $d < k$. But then $y_d = f_{kd}(y_k) = f_{kd}(z_k) = x_k$. So $z = y$ and $U \cap A = \{y\}$. Therefore y is not in A' and $X_\infty$ is scattered.

Note. A preliminary version of this paper was presented to the ICM-86 in Berkeley, California, August 1986.

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# CAPILLARY INSTABILITY OF A STREAMING HOLLOW CYLINDER

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## ABSTRACT

The capillary instability of a hollow jet flow of a streaming liquid surrounding a gas cylinder ( the equilibrium gas pressure is held constant ) subject to its liquid inertia force is presented . A general eigenvalue relation is derived and studied analytically, and the results are confirmed numerically. A comparison is made between the present results and those for the nonhollow-cylinder . The streaming has always a destabilizing influence . The latter attains its maximum value if the basic flow streams in the opposite $z$-direction . In the absence of streaming, the model is stable for all non-axisymmtric modes for all wavelengths while to axisymmtric mode it is stable ( or not ) if the perturbed wavelength is equal to or shorter ( or longer ) than the circumference of the cylinder . The maximum temporal amplification prevailing in the nonhollow case is far lower than of the present model ; this is physically interpreted here .

## 1. INTRODUCTION

Crucial interest in the linear ( [6], [8], [1] ) and nonlinear ( [10], [4, 5], [2] ) hydrodynamic instability of cylindrical jets has been revised due to certain astronomical and industrial problems for many

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applications ranging from the design of sprays to the design of ink jet printers .

In recent years studies on the hollow cylinder stability have been flourishing. In view of its practical applications in the astrophysics science ; Kendall [3] has performed interesting experiments with modern equipment to the annular jet ( gas cylinder submerged in liquid jet ) under the action of surface tension and inertia forces provided that the inertia liquid force is paramount over that of the gas cylinder . Moreover, Kandell [3[ called attention to analytically studying the stability of such models . He mentioned : "it appears that the hollow jet instability and enscapsulation process being studied here have not been repeated in the literature" ( see [3. p. 2086] ) .

The purpose of the present work is to investigate the instability of a streaming hollow cylinder endowed with surface tension and acting upon its liquid inertia force . It is worthwhile to mention here that the phenomenon of the hollow cylinder may occur in the crust of the earth when a gas escaping from below oil layers or in the sea during geological drillings or when air is pumped into the water .

## 2. FORMULATION OF THE PROBLEM AND EIGENVALUE RELATION

Let us consider a uniform gas cylinder of radius $r=R_0$ . The cylinder is immersed in an infinite liquid of density $\rho$ . The liquid has a basic flow of uniform velocity $U$ along the axis of the hollow cylinder. The model ( hollow jet ) is acting upon the capillary ( along the gas-liquid interface ) and inertia forces such that the liquid inertia force is paramount [3] over that of the gas . The effect of gravity

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is negligible for $( gR_0/U^2 ) << 1$. We shall use the cylindrical polar coordinates $(r,\rho,z)$ with the axis of the hollow cylinder coinciding with the z-axis. The fluids are considered to be inviscid and incompressible. The basic equations are those of motion, continuity and that one of surface tension viz [1]

$$(1) \qquad \rho \frac{d\underline{u}}{dt} = \rho \left( \frac{\partial \underline{u}}{\partial t} + ( \underline{u} \cdot \nabla ) \underline{u} \right) = - \nabla \rho$$

$$(2) \qquad div\ \underline{u} = 0$$

$$(3) \qquad P_s = - T ( r_1^{-1} + r_2^{-1} ) = - T\, div\ \underline{n}_s$$

where $\underline{u}$ $P$ and $T$ are the liquid velocity, pressure and surface tension coefficient : $r_1$ and $r_2$ are principle radii of curvature of the cylindrical surface and $\underline{n}_s$ $( = \nabla f / | \nabla f | )$ is a unit outward vector normal (directed from the gas to the liquid) to the surface $f( r,\rho,z ; t ) = 0$.

The equilibrium pressure $P_0$ is given by

$$(4) \qquad P_0 = P_g - ( T/R_0 ) .$$

The term $( - T/R_0 )$ is the surface tension contribution and $P_g$ is a gas constant pressure. The hollow cylinder has to be filled with gas of strong (constant) pressure to insure the equilibrium state snch that $P_g \geqslant T/R_0$ (where equality may occur as limiting case with zero liquid pressure) otherwise the model will collapse towards a hollow cylinder of smaller radius than $R_0$ to reach an equilibrium state.

For small departures from the equilibrium state, let the small departure in the radius of the cylinder be $\zeta$ and proportional to $\exp [ i ( \alpha z + m\rho ) - i\alpha c t ]$ where $\zeta << R_0$.

$$(5) \qquad r = R_0 + \zeta = R_0 + \varepsilon_0 R_0 \exp [ i ( \alpha z + m \rho ) - i \alpha c t ]$$

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where $\varepsilon_0$ is the initial amplitude ( at time $t = 0$ ); $\alpha$ and $m$ are the longitudinal and azimuthal wavenumbers, $c$ is the velocity of light, $\zeta$ represented the surface wave elevation normalized with respect to $R_0$ and measured from the equilibrium position. The linearized equations are

$$(6) \qquad \rho \left( \frac{\partial \underline{u}_1}{\partial t} + U \frac{\partial \underline{u}_1}{\partial z} \right) = - \nabla P_1$$

$$(7) \qquad div\, \underline{u}_1 = 0$$

$$(8) \qquad P_{1s} = ( T/R_0^2 ) ( \zeta = ( \partial^2 \zeta / \partial \rho^2 ) + R_0^2 ( \partial^2 \zeta / \partial z^2 ) ) : \text{ at } r = R_0$$

Consequent to deformation (5) as usual for the stability problems of cylindrical configurations, based on the linear-perturbation technique we assume that any perturbed quantity can be expressed as $[ i( \alpha z + m\rho ) - i\alpha c t ]$ times an amplitude function of $r$. The relevant perturbation equations are solved and the non-singular soiution is given by

$$(9) \qquad \underline{u}_1 = A [ \alpha K_m' ( \alpha r ), \frac{im}{r} K_m ( \alpha r ), i\alpha K_m ( \alpha r ) ]$$

$$exp [ i ( \alpha z + m\rho ) - i \alpha c t ]$$

$$(10) \qquad P_1 = - \rho ( U - c ) u_{1z}.$$

$$(11) \qquad \varepsilon_0^{-1} P_{1s} = ( T/R_0 ) ( 1 - m^2 - \alpha^2 R_0^2 ) exp [ i ( \alpha z + m\rho ) - i\alpha ct ],$$

where A is unspecifed constant and $K_m ( \alpha r )$ is the modified Bessel function of the second kind of the order $m$. The appropriate boundary conditions are : (I) the kinematic condition that the normal component of the fluid velocity must be compatible with the deformed

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surface (5) at $r = R_0$, yields $u_{1r} = \partial\zeta/\partial t + U\,\partial\zeta/\partial z$ from which one gets $A = iR_0 ( U - c ) / K'_m ( \alpha R_0 )$ ; and (II) the kinetic pressure of the fluid must cope with the pressure due to the capillary force at $r = R_0$ . The last condition gives, at once, the following relation :

$$(12) \quad \omega^2 = ( T/\rho R_0^3 ) ( x^2 + m^2 - 1 ) F_m (x)$$

where $\omega = ( U - c ) \alpha$

$$F_m (x) = - x K'_m (x) / K_m (x)$$

and where $x ( = \alpha R_0 )$ is the longitudinal dimensionless wavenumber.

## 3. DISCUSSIONS AND CONCLUSIONS

Equation (12) is the desired eigenvalue relation of a capillary streaming hollow cylinder ( valid for all modes of perturbation $m \geqslant 0$ ) subjecting to its liquid inertia force . It relates the angular frequency $\omega$ ( or rather the temporal amplification $\sigma$ if $\omega$ is imaginary ( $\sigma = - i\omega c$ ) ) with the entity $( T/ \rho R_0^3 )^{-1/2}$ as unit of time, the wavenumber $x$ and $m$ , the second kind modified Bessel function and its derivative ( with respect to its argument ), the characteristic length $R_0$ and the other parameters $T$, $\rho$, $U$ and $c$ of the problem .

In the absence of streaming ( $U = 0$ ) and to axisymmetric disturbances $m = 0$ ; equation (12) reduces to

$$(13) \quad \sigma^2 = ( T/\rho R_0^2 [ x K_1 (x) / K_0 (x) ] ( x^2 - 1 ) .$$

This coincides with dispersion relation obtained by us [7] as the viscosity effect is suppressed there .

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By using the recurrence relation of the modified Bessel function

(14) $2K_m'(x) = -K_{m-1}(x) - K_{m+1}(x)$

and to each non-zero real value of $x$ such that $K_m(x)$ is monotonic decreasing but never negative . One can show that

(15) $F_m(x) > 0$

to all modes of perturbation $m \geqslant 0$ for all ( long and short ) wavelengths and that $F_m(x)$ never changes sign . By using the inequality (15); equation (12) yields that $\omega^2 > 0$ for all $m \neq 0$, but that $\omega^2 < 0$ for $-1 < x < 1$ and $\omega^2 \geqslant 0$ for $x \geqslant 1$ or $x \leqslant -1$ if $m = 0$ . That is exactly the same situation for the nonhollow cylinder (full liquid jet) . It is stable to all nonaxisymmetric disturbances $m \geqslant 1$ for all wavelengths while to $m=0$ it is stable ( or not ) if the perturbed wavelength is equal to or shorter ( or longer ) than the circumference of the jet. From the analytical discussions of eq. (12), one can easily show that the streaming has a destabilizing effect . The latter attains its maximum value if the basic flow streams (for given $U$) in the opposite $z$-direction . The destabilizing character of the streaming is also true even in the presence of different forces [9] . The analytical results are confirmed numerically by using the relation (12) and that of full liquid jet

(6) $\omega^2 = (T/\rho R_0^3)[x I_m'(x)/I_m(x)](x^2 + m^2 - 1)$

( where $\omega$, $T$, $\rho$, $R_0$, $x$ and $m$ have their usual meaning while $I_m(x)$ is the first *kind* modified Bessel function of the order $m$ ) in the computer simulation to axisymmetric mode $m = 0$ and non-axisymmetric modes 1 and 2 .

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The oscillation curves or the hollow cylinder ( for all $x$ values to $m=1$ and 2 and also to $m=0$ if $x \geqslant 1$ ) are little bit higher than those of the full jet in the same domains of stability .

The domain of instability is $0 < x < 1$ to $m=0$ only . The maximum temporal amplification prevailing the nonhollow case is for lower than that of the present model . The maximum modes cf instabillty ( $=0.34$ for full jet and $=0.82$ for hollow cylinder ) give fair feeling for that . This may be due to the fact that the pressure pushes in the same direction as the surface tension does. This reason is not oniy the cause of that since we have used volume conservation and the cylinder is filled with gas of ( constant ) strong equilibrium pressure . However, another reason can be stated if one takes into account the conservatin energy principle . The only source that can deliver energy is the potential energy due to surface tension . The greater instability is due to the fact that the same amount of potential energy can more easily give rise to motions, *i.* e., increasing the kinetic energy according to the energy conservation principle (needing little kinetic energy ) because the hollow cylinder extends radially to infinity ; in comparison with the liquid cylinder ( which ) due to its radial iimitation needs comparatively large speeds in the z-direction in particular for long wavelengths $x << 1$ ) uses more kinetic energy for the same radial deformations .

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# PERTURBATIONS OF PERIGEE AND APOGEE DISTANCE OF EARTH'S SATELLITE ORBITS DUE TO EARTH'S OBLATENESS AND SOLAR RADIATION PRESSURE

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## 1. INTRODUCTION

Levin [1] studied the solar radiation pressure perturbation effect on the satellite orbit and calculated apsidal locations and distances, neglecting the other perturbing forces such as Earth's oblateness . He calculated all his results by taking the sun and the satellite orbit in the equatorial plane and ignored the obliquity of the ecliptic . In this paper we have studied the distances and locations of the perigee and apogee by taking into account the effect of solar radiation pressure and oblateness of the Earth . We have taken the sun and the satellite orbit in the plane of ecliptic . We have considered the problem for a fixed source as well as moving source. Shadow effect of the Earth is also taken into account . In all the cases, the orbit of the satellite is supposed to be circular but in section 4, it is taken to be elliptic . However the eccentricity considered is of the $O(10^{-3})$ . We have taken the perturbing acceleration upto $O(10^{-5})$.

## 2. EQUATIONS OF MOTION

The force exerted by radiation pressure upon a satellite depends on the intensity of the radiation, the presented normal area of the satellite and reflectivity of the surface . The magnitude of the perturbing acceleration on a particle of mass $m$ and cross-section $A_N$ may be written as Levin [1]

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$$K = \frac{(1+\rho)\, s}{C} \times \frac{A_N}{m} \quad cm/sec^2$$

where $\rho$ is the reflectivity, $0 \leqslant \rho \leqslant 1$, $s$ is the solar constant ($\sim 1.35 \times 10^6$ *ergs/cm² sec*) and $C$ is the velocity of light ($\sim 3 \times 10^{10}$ *cm/sec*).

For the sake of simplicity, we assume a nominally circular orbit of the satellite and fixed sun at $x = -\infty$ anywhere in the ecliptic plane. We also assume that the Earth, the sun and the satellite orbit are in the same plane. Further, we neglect the shadow and the reflected radiation by the earth. If $X$ is the position of the satellite which moves in a circular orbit around the Earth ($E$) Fig. 1 ( a, b ) and the

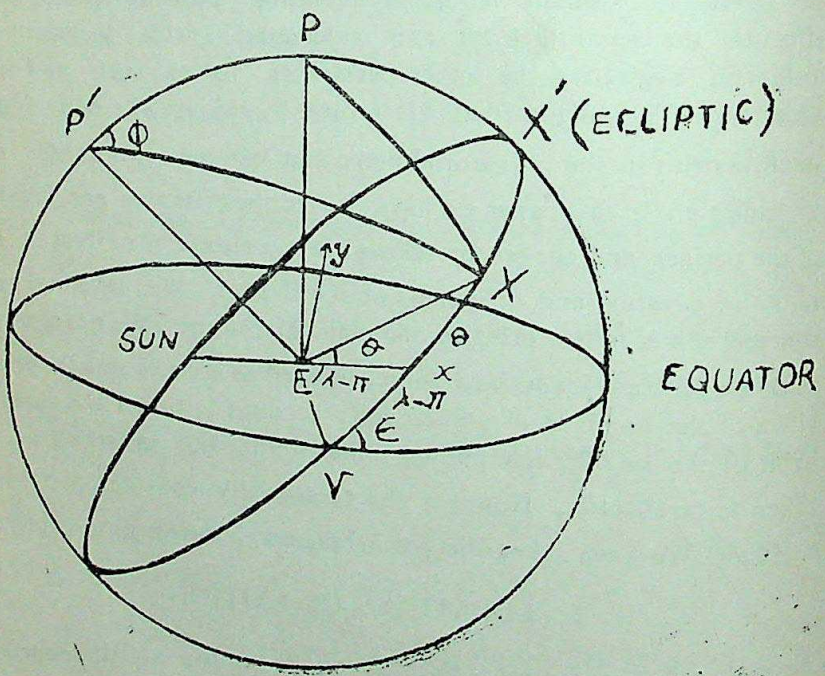


FIG 1 (a)

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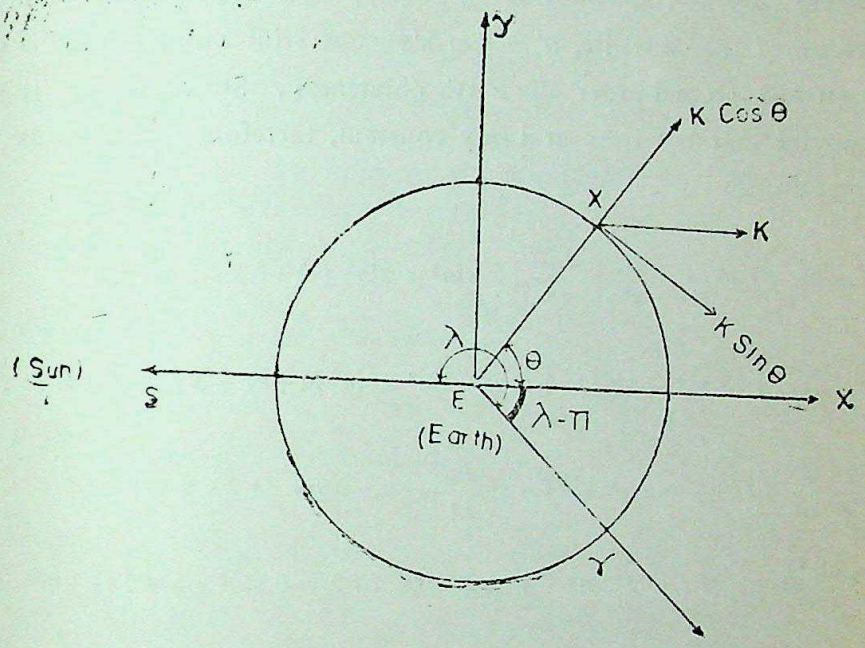


(Fig 1)(b)

angle θ that $EX$ makes with $SE$ produced, the line joining the Sun and the Earth, then the radial and transverse components of the perturbing acceleration are $K_r = K \cos \theta$ ; $K_\theta = -K \sin \theta$. If $\lambda$ is Sun's longitude and $\delta$ the declination of the satellite, we have $\sin \delta = -\sin \varepsilon \sin (\lambda + \theta)$ where $\varepsilon$ is the obliquity .

Again, the potential of the Earth at an external point is given by Sterne [3]

$$V = \frac{\mu}{r} \left[ 1 - \frac{1}{3} \frac{Ja'^2}{r^2} ( 3 \sin^2 \delta - 1 ) + \frac{D}{35} \left( \frac{a'}{r} \right)^4 ( 35 \sin^4 \delta - 30 \sin^2 \delta + 3 ) \right]$$

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where $\mu = Mf$ ; $M = mass$ of the Earth; $f$ = gravity constant, $r$ = radius vector of the satellite, $a'$ = Earth's equatorinl radius, $J$ and $D$ are first and second order of Earth oblateness. Since the sun is not moving that is $\lambda$ is an arbitrary constant, therefore the epuatioas of motion are

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2} + \frac{Ja'^2}{r^4}\left[3\sin^2\varepsilon\sin^2(\lambda+\theta) - 1\right] -$$

$$\frac{\mu Da'^4}{7r^6}\cdot\left[35\sin^4\varepsilon\sin^4(\lambda+\theta) - 30\sin^2\varepsilon\sin^2(\lambda+\theta) + 3\right] + K\cos\theta \quad (1)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = -K\sin\theta - \frac{\mu Ja'^2}{r^4}\sin^2\varepsilon\sin 2(\lambda+\theta) +$$

$$\frac{\mu Da'^4}{7r^6}\left\{(28\sin^4\varepsilon\sin^3(\lambda+\theta)\cos(\lambda+\theta) - 6\sin^2\varepsilon\sin 2(\lambda+\theta)\right\} \quad (2)$$

We take the initial conditions as

$$\theta_0 = -\pi/2;\ r = r_0;\ (r^2\dot{\theta})_0^2 = \mu r_0;\ \dot{r} = 0.$$

Putting $P = r^2\dot{\theta}$ in equation (2), integrating and applying initial conditions, we get $P^2$. If we put $u = 1/r$ and using the value of $P^2$, we get a second order differential equation. We have included terms $O(10^{-5})$. Solving this differential equation applying initial conditions and substituting $u = 1/r$, we get

$$r = a + f(\sin\theta, \cos\theta) + A\theta\sin\theta - B\theta\cos\theta \quad (3)$$

where $A = \frac{KJr_0 a'^2}{\mu}\left\{-3\sin^2\varepsilon\cos 2\lambda - \frac{3}{2}\sin^2\varepsilon + 1\right\} + \frac{3Kr_0^3}{2\mu} > 0,$

$\forall\ \lambda,\ as\ \sin\varepsilon \simeq \frac{2}{5},$ ... (3a)

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$$B = \frac{KJa'^2 r_0}{2\mu} \sin^2 \varepsilon \sin 2\lambda,$$

'$a$' is some constant and $f$ is a function of $\sin\theta$ and $\cos\theta$.

Finally, we find the approximate solution as $\theta_{approx} = n\pi + \alpha$, where $n$ is any positive integer or zero and $\alpha = \tan^{-1}(-A/B)$. Therefore from equation (3), we have

$$r_{approx} = r_0 - \left[ -\frac{1}{3}\frac{Ja'^2}{r_0} \sin^2\varepsilon \sin 2\lambda - \frac{3Kr_0^3}{2\mu} + \right.$$

$$+ \frac{KJr_0a'^2}{2\mu} \sin^2\varepsilon \{ (3+5\cos 2\lambda) - 2 \}.$$

$$- \frac{KJr_0a'^2}{8\mu} \sin^2\varepsilon \{ 3\cos 3\lambda - 4\cos 2\lambda + 2(-\pi + 4\lambda)\sin 2\lambda \} + \frac{6}{7}\frac{Da'^4}{r_0^3} \times$$

$$\times \sin^2\varepsilon \sin 2\lambda - \frac{1}{10}\frac{Da'^4}{r_0^3} \sin^4\varepsilon \{ (\sin 4\lambda + 10\sin 2\lambda \}$$

$$- \frac{2}{3}\frac{J^2a'^4}{r_0^3} \sin^2\varepsilon \sin 2\lambda + \frac{J^2a^4 \sin^4\varepsilon}{r_0^3} \sin 2\lambda +$$

$$\left. + \frac{7J^2a'^4 \sin^4\varepsilon}{30r_0^3} \sin 4\lambda \right] \cos(n\pi + \alpha) -$$

$$- \left[ \frac{Ja'^2}{r_0} \{ (1 - \frac{3}{2}\sin^2\varepsilon - \frac{5}{6}\cos 2\lambda \sin^2\varepsilon \} + \right.$$

$$+ \frac{Da'^4}{7r_0^3} \{ 14\sin^4\varepsilon\cos^4\lambda + 3 - 3\sin^2\varepsilon(\cos 2\lambda + 5) \} +$$

$$\left. + \frac{Da'^4 \sin^4\epsilon}{40r_0^3} \{ -\cos 4\lambda - 20\cos 2\lambda + 45 \} - \right.$$

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$$- \frac{3Kr_0^3}{2\mu} \cdot \frac{\pi}{2} + \frac{KJr_0a'^2}{4\mu} \{ 6\pi \sin^2 \varepsilon \cos 2\lambda + \sin^2 \varepsilon ( 3\pi + \frac{\sin 2\lambda}{2} -$$

$$- 4\lambda \cos 2\lambda ) - 2\pi \} + \frac{4K^2 r_0^5}{\mu^2} - \frac{4}{3} \frac{J^2a'^4 \sin^2 \varepsilon}{r_0^3} \cos 2\lambda +$$

$$+ \frac{J^2a'^4 \sin^4 \varepsilon}{r_0^3} \cos^2 2\lambda + \frac{2J^2a'^4 \sin^4 \varepsilon}{r_0^3} \cos 2\lambda + \frac{1}{10} \frac{J^2a'^4 \sin^4 \varepsilon}{r_0^3} \cos 4\lambda +$$

$$+ \frac{J^2a'^4 \sin^4 \varepsilon}{12r_0^3} \Big] \sin ( n\pi + \alpha ) - \frac{Ja'^2}{r_0} \{ 1 - ( \frac{3}{2} + \cos 2\lambda ) \sin^2 \varepsilon \} -$$

$$- \frac{Da'^4}{7r_0^3} \{ 14 \sin^4 \varepsilon \cos^4 \lambda + 3 - 3 \sin^2 \varepsilon ( 2 \cos 2\lambda + 5 ) \} +$$

$$+ \frac{J^2a'^4 \sin^2 \varepsilon}{r_0^3} \cos 2\lambda - \frac{3J^2a'^4 \sin^4 \varepsilon}{2r_0^3} \cos 2\lambda - \frac{J^2a'^4 \sin^4 \varepsilon}{r_0^3} \cos^2 2\lambda +$$

$$+ \frac{3Kr_0^3}{2\mu} ( n\pi + \alpha ) \sin ( n\pi + \alpha ) + \frac{1}{6} \frac{Ja'^2}{r_0} \sin^2 \varepsilon \cos ( 2n\pi + 2\alpha + 2\lambda )$$

$$- \frac{KJa'^2 r_0}{2\mu} \{ \sin^2 \varepsilon ( 3 + 5 \cos 2\lambda - 2 ) \} ( n\pi + \alpha ) \sin ( n\pi + \alpha ) -$$

$$- \frac{KJr_0a'^2}{8\mu} \sin^2 \varepsilon \{ -\cos (3n\pi + 3\alpha + 2\lambda) + 4 (n\pi + \alpha + 2\lambda) \sin (n\pi + \alpha + 2\lambda) \}$$

$$- \frac{K^2 r_0^5}{\mu^2} \{ 3 - \cos ( 2n\pi + 2\alpha ) \} + \frac{3}{7} \frac{Da'^4}{r_0^3} \sin^2 \varepsilon \cos (2n\pi + 2\alpha + 2\lambda) -$$

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$$+\frac{1}{40}\frac{Da'^4}{r_0^3}\sin^4\varepsilon\{-\cos 4(n\pi+\alpha+\lambda)+20\cos(2n\pi+2\alpha+2\lambda)+45\}-$$

$$-\frac{J^2a'^4\sin^2\varepsilon}{3r_0^3}\cos(2n\pi+2\alpha+2\lambda)+\frac{J^2a'^4\sin^4\varepsilon}{12r_0^3}\cos(2n\pi+2\alpha+4\lambda)+$$

$$-\frac{J^2a'^4\sin^4\varepsilon}{12r_0^3}\cos(2n\pi+2a)+\frac{J^2a^4\sin^4\varepsilon}{2r_0^3}\cos(2n\pi+2\alpha+2\lambda)-$$

$$-\frac{J^2a'^4\sin^4\epsilon}{60r_0^3}\cos(4n\pi+4\alpha+4\lambda).$$

Now in order to study the perigee and apogee distances, we shall consider the following two cases : (i) $B\neq 0$ and (ii) $B=0$, where $\tan\alpha=-A/B$ and $A>0$.

Case (i) (a) Suppose $B>0$, this implies $0<\lambda<\frac{\pi}{2}$ or $\pi<\lambda<\frac{3\pi}{2}$.

As $A>0$, so $\tan\alpha=-A/B<0$; $\frac{\pi}{2}<\alpha<\pi$. We get perigee or apogee according as $n$ is odd or even :

$$r_p=r_{perigee}=a+f\{\sin\alpha,\cos\alpha\}-(n\pi+\alpha(A^2+B^2)^{1/2}$$

$$r_a=r_{apogee}=a+f\{\sin\alpha,\cos\alpha\}+(n\pi+\alpha)(A^2+B^2)^{1/2}$$

(b) Suppose $B<0$, this implies $\frac{\pi}{2}<\lambda<\pi$ or $\frac{3\pi}{2}<\lambda<2\pi$. As $A>0$, so $\tan\alpha=-\frac{A}{B}>0$; $0<\alpha<\frac{\pi}{2}$. Again we get perigee or appgee according as $n$ is odd or even :

$$r_p=a+f\{\sin\alpha,\cos\alpha\}-(n\pi+\alpha)(A^2+B^2)^{1/2}.$$

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Similarly apogee can be calculated .

Case (ii) (a) Suppose $\varepsilon \neq 0, \lambda = \pi \Rightarrow B = 0$, therefore $\theta = n\pi - \frac{\pi}{2}$, that is the case when sun is fixed in the line of vernal equinox at $x = -\infty$ . We get perigee or apogee according as $n$ is even or odd as

$$r_p = r_0 - \frac{Kr_0}{2\mu} (3r_0^2 + 2Ja'^2) \times 2m\pi + \frac{KJr_0a^2}{\mu} \cdot \frac{9}{2} \sin^2 \varepsilon \times 2m\pi$$

Similarly apogee distance can be calculated .

(b) Suppose $\varepsilon = 0$ than $B = 0$ and there $\theta = n\pi - \pi/2$ that is, the case when the Sun is fixed anywhere in the equatorial plane at $x = -\infty$. We get perigee or apogee according as $n$ is even or odd as

$r_p = r_0 - \frac{Kr_0}{2\mu} (3r_0^2 + 2Ja'^2) \times 2m\pi$, where $m$ is zero or positive integer .

Similarly apogee can be calculated .

In both the above cases, we note that the rate of decrease of perigee and the rate of increase of apogee with each revolution is the same. therefore the semi-major axis $\frac{1}{2}$( perigee + apogee ) remains fixed and therefoae period of revolution is approximately constant .

## 3. EFFECT OF SHADOW

Again, we shall assume that the satellite moves in a circle and the radiation source is fixed as $x = -\infty$ . We further assume that the Earth, the Sun and the satellite orbit are in the same plane .

Let the shadow region be defined by ( Fig. 2 )

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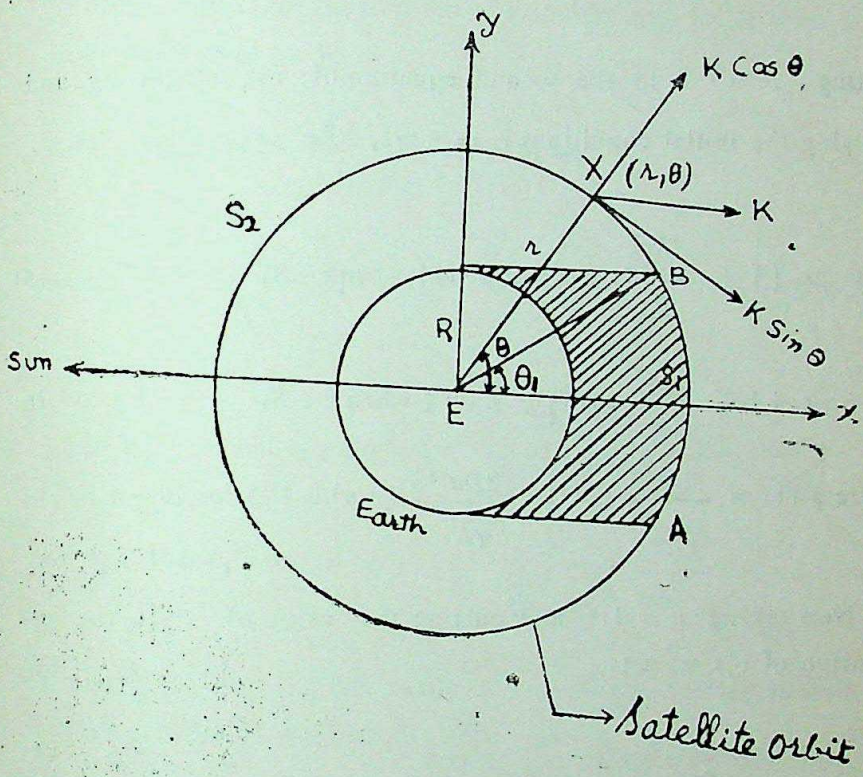


(FIG - 2)

$S_1 : -\theta_1 \leqslant \theta \leqslant \theta_1; S_2 : \theta_1 < \theta < 2\pi - \theta_1$

where $\theta_1 = \cos^{-1} \sqrt{(1-\sigma^2)}$, $R/r = \sigma$ and $R$=the radius of the Earth;

Equations of motion are

$$\left.\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= F_r + K\cos\theta \\ \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) &= F_\theta - K\sin\theta \end{aligned}\right\} \theta \in S_2 \qquad \left.\begin{aligned} &= F_r \\ &= F_\theta \end{aligned}\right\} \theta \in S_1 \qquad (4)$$

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where the values of $F_r$ and $F_\theta$ can be written from equations (1) and (2).

Putting $P = r^2\dot{\theta}$ in the second equation of (4), integrating and applying the initial conditions $\theta_0 = -\pi/2$, $P_0^2 = \mu r_0$, we get

$$P^2 = \mu r_0 \left[ 1 + \frac{2Kr_0^2}{\mu} \cos\theta + \phi(\theta) \right] \text{ when } \theta \in S_2 \tag{5}$$

$$P^2 = \mu r_0 \left[ 1 + \frac{2Kr_0^2}{\mu} \cos\theta_1 + \phi(\theta) \right] \text{ when } \theta \in S_1 \tag{6}$$

where $\phi(\theta) = \frac{2Ja'^2}{r_0^2} \cos^2\theta + \frac{2Da'^2}{7r_0^4}(7\sin^4\theta + 3\cos 2\theta - 4)$.

Now taking $u = 1/r$ and putting the value of $P^2$ in the first equation of (4), we get

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{r_0} - f(\theta) \tag{7}$$

where $f(\theta) = \frac{3Kr_0}{\mu}\cos\theta - \frac{4KJa'^2}{\mu r_0}(1+\cos^2\theta)\cos\theta - \frac{6K^2r_0^3}{\mu^2}\cos^2\theta + P(\theta), \theta \in S_2$

and similarly the value of $f(\theta)$ can be calculated when $\theta \in S_1$,

where $P(\theta) = \frac{Ja'^2}{r_0^3}(1+\sin^2\theta) - \frac{Da'^4}{7r_0^5}(21\sin^4\theta - 24\sin^2\theta + 8) -$

$$- \frac{2J^2a'^4}{r_0^5}\cos^2\theta(2 - \cos^2\theta).$$

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In order to find out the solution of (7), $f(\theta)$ may be expressed by its Fourier series in the form

$$f(\theta) = A_0 + A_1 \cos\theta + \sum_{m=2}^{\infty} A_m \cos m\theta$$

where $A_0 = \frac{Kr_0}{\mu\pi}\{2\theta_1 \cos\theta_1 - 3\sin\theta_1\} + \frac{3Ja'^2}{2r_0^3} + \frac{KJa'^2}{\pi\mu r_0}(14\sin\theta_1 +$

$\frac{2}{3}\sin 3\theta_1 - 5\theta_1\cos\theta_1 - \sin\theta_1\cos^2\theta_1) - \frac{K^2r_0^3}{\pi\mu^2}\{3(\pi-\theta_1) -$

$\cos\theta_1(3\sin\theta_1 - 4\theta_1\cos\theta_1)\} - \frac{31}{56}\frac{Da'^4}{r_0^5} - \frac{5}{4}\frac{J^2a'^4}{r_0^5}$ (8)

Similarly $A_1$ and $A_m$ can easily be calculated. Solving equation (7) applying initial conditions, we get

$$u = \frac{1}{r_0} - A_0 - \left(\frac{A_1}{2} + S_0\right)\cos\theta - (A_0 - S_1)\sin\theta - \frac{A_1}{2}(\theta + \pi/2)\sin\theta +$$

$$\sum_{m=2}^{\infty} \frac{A_m \cos m\theta}{(m^2 - 1)} \quad (9)$$

where $S_0 = \sum_{m=1}^{\infty} \frac{(2m+1)(-1)^{m+1} A_{2m+1}}{(2m+1)^2 - 1}$, $S_1 = \sum_{m=1}^{\infty} \frac{(-1)^m A_{2m}}{4m^2 - 1}$

The solution of (9) is $\theta_{apse} \simeq 2n\pi \pm \pi/2$. From equation (9). we have

$$r_{perigee} = r_0\left[1 - 2n\pi.\frac{3Kr_0^2}{2}\left\{1 - \frac{\theta_1}{\pi} + \frac{\sin 2\theta_1}{6\pi}\right\} + 2n\pi.\frac{7KJa'^2}{2\mu} \times\right.$$

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$$\{1 - \frac{\theta_1}{\pi} - \frac{4}{7\pi} \sin 2\theta_1 - \frac{1}{28\pi} \sin 4\theta_1 + \frac{\cos \theta_1}{7\pi} (11 \sin \theta_1 + \frac{1}{3} \sin 3\theta_1)\} - 2n\pi . \frac{K^2 r_0^4}{2\pi\mu^2} \{9 \sin\theta_1 + \sin 3\theta_1 - 8 \cos^2 \theta_1 \sin \theta_1)\}] \quad (10)$$

and $r_{apogee} = r_0 (1 + 2r_0 (A_0 - S_1) + \frac{A_1 r_0}{2} (2n\pi + \pi)]$.

For a low to medium altitude satellite, $\theta_1$ is large and therefore $\frac{\theta_1}{\pi}$ is the dominant term in (10). Now as $\frac{\theta_1}{\pi}$ is the portion of the circumference in shadow, we note that the effect of shadow is to diminish the height of the perigee by an amount equal to the fraction of the path which is in shadow. Further, for a very distant satellite $\theta_1 \to 0$ and therefore the effect of shadow is almost negligible.

## 4. PERIGEE AND APOGEE DISTANCES FOR ELLIPTIC ORBIT

In this section, we shall try to find apsidal distances and their angular positions for elliptic case taking the radiation source as fixed. The eccentricity considered is of the $O(10^{-3})$.

The equations of motion are the same as given in (4). Putting $P = r^2 \dot{\theta}$ in these equations and using $r = \frac{l}{1 + e \cos(\theta - \Psi)}$, where $\Psi$ is the initial angular position of perigee and $l = a(1 - e^2) = P_0^2/\mu$ is the semi-latus rectum of the ellipse and after applying initial conditions $\theta_0 = -\pi/2$, $P^2 = P_0^2 = \mu l$; we get the value of $P^2$. As before let $u_0 = 1/r$ and putting the value of $P^2$ in (4) and solving it and applying initial conditions $\theta_0 = -\pi/2$, $u = u_0 = \{1 - e \sin \Psi\}/a(1 - e^2)\}$; $(du/d\theta)_0 = (e \cos \Psi)/a(1 - e^2))$, we obtain

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$$u=\frac{1+e\cos(\theta-\Psi)}{l}-\frac{3Kl}{2\mu}\{\cos\theta+(\theta+\pi/2)\sin\theta\}-\frac{3Kle}{2\mu}\{(\pi\sin\Psi-\cos\Psi)+$$

$$+2\theta\sin\Psi+\frac{1}{3}\cos(2\theta-\Psi)-\frac{4}{3}\sin(\theta+\Psi)\Big\}-\frac{Ja'^2}{6l^3}(8\sin\theta+\cos 2\theta+9)+$$

$$+\frac{Da'^4}{280l^3}(128\sin\theta-7\cos 4\theta+20\cos 2\theta+155)+\frac{KJa'^2}{8\mu l}(25\cos\theta-\cos 3\theta+$$

$$+28(\pi/2+\theta)\sin\theta\}+\frac{K^2l^3}{\mu^2}(4\sin\theta-\cos 2\theta+3) \qquad (11)$$

We get $\theta_{apse}\simeq 2n\pi\pm\pi/2$. From equation (11), we get the perigee distance as

$$V_{perigee}=a(1+e\,siu\,\Psi)-\frac{3Ka_3}{\mu}(\tfrac{1}{2}-e\sin\Psi)\,2n\pi+\frac{7KJa'^2}{2\mu}.\,2n\pi a.$$

Similarly, we can obtain apogee by putting $\theta=2n\pi+\pi/2$ in equation (11).

However, perigee is initially located at $\theta=\Psi$, but for small eccentricity after a moderate number of revolutions, it is shifted from $\Psi$ to $(-\pi/2+2n\pi)$. And if the eccentricity is large the motion is much slower and the initial perigee location tends to persist.

## 5. MOVING SOURCE

If the Sun moves from its initial direction at a constant angular rate $N'$, then the radial and transverse components of the perturbing acceleration due to solar pressure at any time $t$ are

$$K_r=K\cos(\theta-N't)\;;\;K_\theta=-K\sin(\theta-N't).$$

As before, we can write thc equations of motion and putting $r^2\dot{\theta}=P$ in the second equation, integrating and taking tha initial conditions

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as $\theta_0 = -\pi/2$, $r = r_0$, $(r^2\dot{\theta})_0^2 = \mu r_0$, $\dot{r} = 0$, we get the value of $P^2$.

We note that $\dot{\theta} = (\frac{\mu}{r_0^3})^{\frac{1}{2}} = constant = N$ ( say ), then

$Nt = \theta + \pi/2$.

Also, $N't = N'/N\,(Nt) = N'/N\,(\theta + \pi/2)$.

Now if we put $u = 1/r$ in the first equation of motion and use the value of $P^2$, we get a second order differential equation. solving it and taking into account that $N'/N << 1$, we have

$$U \simeq C + f(\sin\theta, \cos\theta) + \lambda_3 \cos(3\theta - N't) + \lambda_4 \cos 2(\theta - N't) + N/N' \{-(\lambda_1 + \lambda_2)\cos\theta + \lambda_1 \cos(\theta - N't) + \lambda_2 \cos(\theta + N't) \quad (12)$$

where

$$C \simeq \frac{1}{r_0} + \frac{Ja'^2}{r_0^3}\left(1 - \frac{5}{2}\sin^2\varepsilon\right) + \frac{Da'^4}{7r_0^5}\left(14\sin^4\varepsilon - 21\sin^2\varepsilon + 3\right) +$$

$$\frac{9}{8}\frac{Da'^4}{r_0^5}\sin^4\varepsilon + \frac{3K^2 r_0^3}{\mu^2} + \frac{J^2a'^4\sin^2\varepsilon}{2r_0^5}\left(\frac{9}{4}\sin^2\varepsilon - 2\right),$$

$$f(\sin\theta, \cos\theta) \simeq \lambda.\ N/N' \cos\theta + \{\frac{Ja'^2}{r_0^3}\left(1 - \frac{7}{3}\sin^2\varepsilon\right) +$$

$$\frac{Da'^4}{35r_0^5}\left(91\sin^4\varepsilon - 90\sin^2\varepsilon + 15\right) + \frac{4K^2r_0^3}{\mu^2} + \frac{J^2a'^4\sin^2\varepsilon}{120r_0^5}$$

$$(217\sin^2\varepsilon - 160)\}\sin\theta -$$

$$- \frac{Ja'^2\sin^2\varepsilon}{6r_0^3}\cos 2\theta - \frac{3}{7}\cdot\frac{Da'^4}{r_0^5}\sin^2\varepsilon\cos 2\theta + \frac{Da'^4\sin^4\varepsilon}{40r_0^5}\left(-\cos 4\theta +\right.$$

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$$+ 20 \cos 2\theta ) - \frac{J^2 a'^4 \sin^2 \varepsilon}{3r_0^5} ( 2 \sin^2 \varepsilon - 1 ) \cos 2\theta + \frac{J^2 a^4 \sin^4 \varepsilon}{60 r_0^5} \cos 4\theta,$$

$$\lambda_1 = - \frac{3Kr_0}{2\mu} + \frac{KJa'^2}{\mu r_0} ( 4 \sin^2 \varepsilon - 1 ) ; \lambda_2 = - \frac{KJa'^2 \sin^2 \varepsilon}{2\mu r_0} ,$$

$$\lambda_3 = - \frac{KJa'^2 \sin^2 \varepsilon}{8\mu r_0}, \lambda_4 = \frac{3K^2 r_0^3}{\mu 2}$$

and $$\lambda = \frac{3Kr_0}{2\mu} \cdot \frac{N}{N'} - \frac{KJa'^2 ( 4 \sin^2 \varepsilon - 1 )}{\mu r_0} \cdot \frac{N}{N'} + \frac{KJa^2 \sin^2 \varepsilon}{2\mu r_0} \cdot \frac{N}{N'} .$$

we find the approximate solution of equation (12) is

$$\tan \theta = - p \cot \frac{N t}{2} \text{ where } p = \frac{( \lambda_1 - \lambda_2 )}{( \lambda_1 + \lambda_3 )} > 0 \qquad (13)$$

Taking $\sin \varepsilon \simeq \frac{2}{5}$, we have $P \simeq ( 1 - \beta )$. where $\beta = \frac{8}{75} \cdot \frac{Ja'^2}{r^2}$. Here, it is to be noted that $P$ depends upon oblateness and $\beta$ a small quantity ( $\beta << 1$ ). Also when $J = 0$, $\beta = 0$ and therfore $P = 1$, the solution (13) becomes $\theta = n\pi + \frac{\pi}{2} + \frac{N't}{2}$. This confirm the findings of Levin [1] . Now taking the solurion (13), using $u = 1/r$ and taking into account that $\frac{N}{N'} << 1$, we obtain

$$r_{apse} \simeq r_0 [ 1 \mp \frac{3Kr_0^2}{\mu} \frac{N}{N'} \sin \frac{N't}{2} ( 1 + \beta \cos^2 \frac{N't}{2} \pm \frac{Ja'^2}{r_0^2} (1 - \frac{7}{2} \sin^2 \varepsilon) \times$$

$$\times \cos \frac{N't}{2} ( 1 - \beta \sin^2 \frac{N't}{2} ) - \frac{Ja'^2}{r_0^2} ( 1 - \frac{5}{2} \sin^2 \varepsilon ) +$$

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$$+\frac{Ja'^2 \sin^2 \epsilon}{6r_0^2} (\beta \sin^2 N't - \cos N't)$$

$$\pm\frac{Da'^4}{35r_0^4} (91 \sin^4\varepsilon - 90 \sin^2 \varepsilon + 15) \cos \frac{N't}{2} (1 - \beta \sin^2 \frac{N't}{2}) -$$

$$- \frac{Da'^4}{56r_0^4} (175 \sin^4 \varepsilon - 168 \sin^2 \varepsilon + 24) + \frac{3Da'^4}{7r_0^4} \sin^2\varepsilon (\beta \sin^2 N't - \cos N't)$$

$$- \frac{Da'^4 \sin^4 \epsilon}{40r_0^4} \{ -2 (\cos^2 N't - \beta \sin N't \sin 2N't) + 1 +$$

$$20 (\beta \sin^2 N't - \cos N't) \} -$$

$$\frac{3K^2r_0^4}{\mu^2} \mp \frac{3K^2r_0^4}{\mu^2} \cos \frac{N't}{2} (1 - \beta \sin^2 \frac{N't}{2}) + \frac{3Kr_0^4}{\mu^2} (\beta \sin^2 N't + \cos N't) -$$

$$\frac{J^2a'^4 \sin^2 \varepsilon}{2r_0^4} (\frac{9}{4} \sin^2 \epsilon - 2) + \frac{J^2a'^4 \sin^2 \varepsilon (2\sin^2\varepsilon - 1)}{3r_0^4} (\beta \sin^2 N't - \cos N't)$$

$$+ \frac{J^2a'^4 \sin^4 \varepsilon}{60r_0^4} \{ 1 - 2 (\cos^2 N't - \beta \sin N't \sin N't) \mp \frac{KJa'^2 \sin^2 \epsilon}{8\mu}$$

$$\times \{ \sin \frac{N't}{2} + 3\beta \cos \frac{N't}{2} \sin \frac{N't}{2} (\cos \frac{3N't}{2} + 2 \sin \frac{N't}{2} \sin N't) \}$$

$$\mp \{ - \frac{KJa'^2 (4 \sin^2 \varepsilon - 1)}{\mu} + \frac{KJa'^2\beta}{\mu} (\frac{9}{2} \sin^2 \varepsilon - 1) \cos^2 \frac{N't}{2} \sin \frac{N't}{2}$$

$$+ \frac{KJa'^2}{\mu} (1 - \frac{7}{2} \sin^2 \varepsilon) \sin \frac{N't}{2} - \frac{KJa'^2 \sin^2 \epsilon}{2\mu} (\sin \frac{3N't}{2} +$$

$$+ \frac{\beta}{2} \sin N't \cos \frac{3N't}{2}) \} ]$$

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The upper sign corresponds to perigee and the lower sign to apogee. If we take terms only upto the order of 0 ($10^{-3}$), we have

$$r_{\text{perigee}} \simeq r_0 [\, 1 - r_s - r_E \,], \text{ where } r_s = \frac{3Kr_0^2}{\mu} \frac{N}{N'} \sin \frac{N't}{2},$$

$$r_E = \frac{4}{75} \frac{Ja'^2}{r_0^2} (\, 1 - x \,) \left(\frac{43}{4} - x\right); x = \cos \frac{N't}{2} << 1, 0 \leqslant t < \infty,$$

Since $x < 1$, $r_E > 0$ for all values of $t$ except when $x = \cos \frac{N't}{2} = 1$ or $\frac{43}{4}$. In this case $r_E = 0$. This means that the perigee distance is further decreased due to oblateness. Moreover $r_E$ is a function of time. The greatest value is 94/75, $Ja'^2/r_0^2$ which is attained after a complete revolution of the Sun that is when $N't = 2\pi$. In the second revolution, the value $r_E$ starts decreasing and it becomes zero after the second complete revolution that is when $N't = 4\pi$ The tarms $r_s$ due to solar radiation pressure indicates that the perigee distance is decreased and its greatest value is $3Kr_0^2 / \mu \cdot \frac{N}{N'}$ which is obtained after 1/2 revolution of the Sun. Both $r_s$ and $r_E$ are periodic functions with period $T$ and $2T$ respectively. Thts we see that the effects of moving source and the oblateness of the Earth are to change the linear decay of perigee into a periodic fluctuation.

Tne above results tally with those of Levin when $\varepsilon = 0$ and $J = 0$.

## 5. CONCLUSION

When the source is taken fixed, Levin proved that the perigee is fixed at $-\pi/2$, but the inclusion of oblateness results in the

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shifting of perigee from $-\pi/2$ to $\pi+\alpha$ ( $\alpha=tan^{-1}(-A/B)$ $A$, $B$ defined by equation (3$a$). The efftect of oblateness is to decrease further the perigee distance or to increase the apogee distance. The effect of shadow is to diminish the height of the perigee by an amount equal to the fraction of the path which is in shadow and for a distant satellite, the effect of shadow is negligble. For small eccentricity the perigee, though initially located at $\theta=\Psi$, yet after a moderate number of revolutions is shifted from $\Psi$ to $(-\pi/2+2n\pi)$ and for large eccentricity the motion is much slower and the initial perigee location tends to persist .

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# SERIES OF SQUARES OF LEGENDRE OF THE SECOND KIND

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*( Received : January 13, 1988 )*

## ABSTRACT

A general result of Bailey is employed to obtain new summation formulae involving squares of the Legendre function of the second kind . Closed - form expressions of certain generalised hypergeometric functions are used in the analysis .

## 1. INTRODUCTION

The author recently required a closed-form summation of a particular series of squares of the Legendre function of the second kind $Q_{\nu+n}(x)$ . In spite of the importance of this function in potential theory, there seems to be a gap in the literature with regard to series of the type

$$\sum_{n=0}^{\infty} C_n \left[ Q_{\nu+n}(x) \right]^2, \tag{1.1}$$

where $C_n$ is a general coefficient. In order to tackle this problem, we note that

$$Q_{\nu+n}(x) = \sqrt{\pi}\,\Gamma(1+\nu+n) / \Gamma(3/2+\nu+n) \left[ 2(x-1) \right]^{-\nu-n-1}$$

$$\times {}_2F_1\left( 1+\nu+n,\ 1+\nu+n;\ 2+2\nu+2n;\ \frac{2}{1-x} \right). \tag{1.2}$$

( See Erdélyi *et al.* [ 2, p. 132 ] . )

Slater [ 3, p, 80 ] has given the relation

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$$ {}_2F_1(a, b; c; x)\, {}_2F_1(a, c-b; c; x) $$

$$ = (1-x)^{-a}\, {}_4F_3\left(a, b, c-a, c-b; c, c/2, c/2+1/2; \frac{x^2}{4(x-1)}\right), \qquad (1.3) $$

and if the parameters are suitably specialised, we have

$$ [Q_{\nu+n}(x)]^2 = \frac{\pi\,\Gamma(1+\nu+n)\,\Gamma(1+\nu+n)}{\Gamma(3/2+\nu+n)\,\Gamma(3/2+\nu+n)} [4(x^2-1)]^{-\nu-n-1} $$

$$ \times {}_3F_2\left(1+\nu+n, 1+\nu+n, 1+\nu+n; 2+2\nu+2n, 3/2+\nu+n; \frac{1}{1-x^2}\right), \quad (1.4) $$

which is a convenient result for the present purpose,

## 2. A GENERAL RESULT

W. N. Bailey [1] gave the following important result on the manipulation of series :

If $$ \beta_n = \sum_{r=0}^{n} \alpha_r\, u_{n-r}\, v_{n+r} \qquad (2.1) $$

$$ \gamma_n = \sum_{r=n}^{\infty} \delta_r\, u_{r-n}\, v_{r+n}, \qquad (2.2) $$

where $\alpha_r$, $\delta_r$, $u_r$ and $v_r$ are functions of $r$ only, such that the series $\gamma_n$ exists, then

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \qquad (2.3) $$

provided that all the series concerned are either convergent or terminating.

If we let

$$ u_r = 1/r!,\ v_r = 1/(2+2\nu,r),\ \delta_r = [(1+\nu,r)]^3/(3/2+\nu,r)(1-x^2)^{-r} $$

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and

$$a_r = \frac{((a), r)\, y^r}{((h), r)\, r!}, \tag{2.4}$$

where

$$(a, r) = a(a+1)(a+2)\ldots(a+r-1) = \Gamma(a+r)/\Gamma(a),\ (a,0) = 1 \tag{2.5}$$

and

$$((a), r) = (a_1, r)(a_2, r)\ldots(a_A, r), \tag{2.6}$$

then, after some reduction, it is found that

$$\sum_{n=0}^{\infty} \frac{((a), n)\, y^n}{((h), n)\, n!} [Q_{\nu+n}(x)]^2$$

$$= \pi [\Gamma(1+\nu)/\Gamma(3/2+\nu)]^2 [4(x^2-1)]^{-\nu-1}$$

$$\times \sum_{n=0}^{\infty} \frac{(1+\nu, n)(1+\nu, n)(1+\nu, n)}{(2+2\nu, n)(3/2+\nu, n)\, n!} (1-x^2)^{-n} A+1\ F_{H+1}\left(\begin{matrix}(a), & -n \\ (h), & 2+2\nu+n\end{matrix}; -y\right). \tag{2.7}$$

Useful results follow from this formula whenever the generalised hypergeometric function on the right of (2.7) is summable as a gamma product. All the summation formulae used below are to be found in Slater [3, p. 243, Appendix III].

## 3. SPECIAL CASES

Certain representative instances are now worked out in some detail, and the remaining formulae given are presented without proof. Let the inner series on the right of (2.7) take the form

$${}_2E_1(a, -n; 2+2\nu+n; -y), \tag{3.1}$$

If we let $y = -1$, then Vandermonde's theorem may be applied to the function (3.1) which becomes

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$$\frac{(2+2\nu-a+n,n)}{(2+2\nu+n,n)} = \frac{(2+2\nu-a,2n)\,(2+2\nu,n)}{(2+2\nu-a,n)\,(2+2\nu,2n)}. \qquad (3.2)$$

The general result (2.7) may then be written

$$\sum_{n=0}^{\infty} \frac{(a,n)\,(-1)^n}{n!} \left[ Q_{\nu+n}(x) \right]^2$$

$$= \pi \left[ \Gamma(1+\nu)/\Gamma(3/2+\nu) \right]^4 \left[ 4(x^2-1) \right]^{-\nu-1}$$

$$\times {}_4F_3\left( \begin{matrix} 1+\nu, 1+\nu, 1+\nu-a/2, 3/2+\nu-a/2; \\ 3/2+\nu, 3/2+\nu, 2+2\nu-a \end{matrix} ; \frac{1}{1-x^2} \right). \qquad (3.3)$$

The series (3.1) may also be summed by Kummer's theorem if we put $y=1$ and $a=1+2\nu$, when

$${}_2F_1(1+2\nu, -n; 2+2\nu+n; -1) = (2+2\nu,n)/(3/2+\nu,n). \qquad (3.4)$$

On substituting this expression into (2.7), we see that

$$\sum_{n=0}^{\infty} \frac{(1+2\nu,n)}{n!} \left[ Q_{\nu+n}(x) \right]^2$$

$$= \pi \left[ \Gamma(1+\nu)/\Gamma(3/2+\nu) \right]^2 \left[ 4(x^2-1) \right]^{-\nu-1}$$

$$\times {}_3F_2(1+\nu, 1+\nu, 1+\nu; 3/2+\nu, 3/2+\nu; 1/(1-x^2)). \qquad (3.5)$$

The formula

$${}_5F_4\left( \begin{matrix} a, 1+a/2, b, c, -n \\ a/2, 1+a-b, 1+a-c, 1+a+n \end{matrix} ; 1 \right)$$

$$= \frac{(1+a,n)\,(1+a-b-c,n)}{(1+a-b,n)\,(1+a-c,n)}, \qquad (3.6)$$

( Slater [ 3. p. 244 ] ) is one of the more complicated results which may be applied to sum the hypergemetric function on the right of

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(2.7) . Put $y = -1$, $a_1 = 1+2\nu$, $a_2 = 3/2+\nu$, $a_3 = b$, $a_4 = c$, $h_1 = 1/2+\nu$, $h_2 = 2+2\nu-b$ and $h_3 = 2+2\nu-c$, when the hypergeometric function under consideration becomes equal to

$$\frac{(2+2\nu,n)\ (2+2\nu-b-c,n)}{(2+2\nu-b,n)\ (2+2\nu-c,n)} \tag{3.7}$$

and we obtain the formula

$$\sum_{n=0}^{\infty} \frac{(1+2\nu,n)\ (3/2+\nu,n)\ (b,n)\ (c,n)\ (-1)^n}{(1/2+\nu,n)\ (2+2\nu-b,n)\ (2+2\nu-c,n)\ n!} [\ Q_{\nu+n}(x)\ ]^2$$

$$= \pi\ [\Gamma(1+\nu)\ /\ \Gamma(3/2+\nu)\ ]^2\ [\ 4\ (x^2-1)\ ]^{-\nu-1}$$

$$\times\ {}_4F_3\left(\frac{1+\nu,\ 1+\nu,\ 1+\nu,\ 2+2\nu-b-c\ ;}{3/2+\nu,\ 2+2\nu-b,\ 2+2\nu-c\ ;}\ 1/(1-x^2)\right). \tag{3.8}$$

As an example of a further special case, if we put $c = 2+2\nu-b$, the ${}_4F_3$ series on the right reduces to unity, giving the rather curious result :

$$\sum_{n=0}^{\infty} \frac{(1+2\nu,n\ (3/2+\nu,n)\ (-1)^n}{(1/2+\nu,n)\ n!} [\ Q_{\nu+n}(x)\ ]^2 = \pi\ [\ \Gamma(1+\nu)/\Gamma(3/2+\nu)\ ]^2$$

$$[\ 4\ (x^2-1)\ ]^{-\nu-1}\ . \tag{3.9}$$

Other similar formulas may be obtained by using the summation theorems for the gereralised hypergeometric series, which are to be found in Slater [3, Appendix III ] . These new expressions are listed below and the corresponding summation theorem is indicated for each case .

$$\sum_{n=0}^{\infty} \frac{(1+2\nu,n)\ (b,n)\ (-1)^n}{(2+2\nu-b,n)\ n!} [\ Q_{\nu+n}(x)\ ]^2$$

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$$= \pi \left[ \Gamma (1+\nu) / \Gamma (3/2+\nu) \right]^2 \left[ 4 (x^2-1) \right]^{-\nu-1} \qquad \text{(III.9)}$$

$$\times {}_4F_3 \left( \begin{matrix} 1+\nu, 1+\nu, 1+\nu, 3/2+\nu-b \\ 3/2+\nu, 3/2+\nu, 2+2\nu-b \end{matrix} ; \frac{1}{1-x^2} \right). \qquad (3.10)$$

$$\sum_{n=0}^{\infty} \frac{(1+2\nu,n)\,(3/2+\nu,n)}{(1/2+\nu,n)\;n!} \left[ Q_{\nu+n}(x) \right]^2 \qquad \text{(III.25)}$$

$$= \pi \left[ \Gamma(1+\nu) \,/\, \Gamma(3/2+\nu) \right]^2 \left[ 4 (x^2-1) \right]^{-\nu-1}$$

$$\times {}_2F_1 \left( \begin{matrix} 1+\nu, 1+\nu \\ 3/2+\nu \end{matrix} ; \frac{1}{1-x^2} \right), \qquad (3.11)$$

$$\sum_{n=0}^{\infty} \frac{(1+2\nu,n)\,(3/2+\nu,n)\,(b,n)}{(1/2+\nu,n)\,(2+2\nu-b,n)\;n!} \left[ Q_{\nu+n}(x) \right]^2$$

$$= \pi \left[ \Gamma(1+\nu)/\Gamma(3/2+\nu) \right]^2 \left[ 4 (x^2-1) \right]^{-\nu-1}. \qquad \text{(III.11)}$$

$$\times {}_3F_2 \left(1+\nu, 1+\nu, 1+\nu; 3/2+\nu, 2+2\nu-b; 1/(1-x^2)\right) \qquad (3.12)$$

and

$$\sum_{n=0}^{\infty} \frac{(1+2\nu,n)\,(3/2+\nu,n)\,(b,n)\,(-1)^n}{(1/2+\nu,n)\,(2+2\nu-b,n)\;n!} \left[ Q_{\nu+n}(x) \right]^2$$

$$= \pi \left[ \Gamma(1+\nu) \,/\, \Gamma(3/2+\nu) \right]^2 \left[4 (x^2-1) \right]^{-\nu-1} \qquad \text{(III.26)}$$

$$\times {}_4F_3 \left( \begin{matrix} 1+\nu, 1+\nu, 1+\nu, 3/2+\nu-b \\ 3/2+\nu, 2+2\nu-b, 2+2\nu-b \end{matrix} ; 1/(1-x^2) \right). \qquad (3.13)$$

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# APPLICATIONS OF FIXED POINT THEOREMS

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Several interesting results using fixed point theory are given in Approximation Theory. An excellent reference is Cheney [3]. For a survey paper one is referred to [9]

The following theorem due to Brosowski [1] has been a basic important result .

Let $f$ be a contractive linear operator of a normed liner space $X$. Let $C$ be an $f$-invariant subset of $X$ and $x$ an $f$ - invariant point. . If the set of best $C$ approximants to x is nonempty, cnmpact and convex, then it contains an $f$ - invariant point .

In [10] Singh observed that the lincarity of $f$ and the convexity of the set of best $C$-approximants to x, can be relaxed .

Subrahmanyam [12], S. P. Singh [10], [11] : and K. L. Singh [9] deal with fixed point theorems of nonexpansive mappings. Carbone [2] and Rao and Mariadoss [7] deal with generalized non-expansive mappings, Sahab, Khan and Sessa [8] extended the result of [10] in a different direction whereas Hicks and Humphries relaxed the boundary condition in their paper [5] .

The purpose of this paper is to give further results unifying earlier known results.

We need the following

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Let $f : X \to X$. Then $f$ is called nonexpansive mapping if

$\|fx - fy\| \leqslant \|x-y\|$ for all $x, y \in X$.

If $\|fx-p\| \leqslant \|x-p\|$ for all $x \in X$ and $p \in F(f)$, then $f$ is called a quasi nonexpansive map ($F(f)$ stands for the set of fixed points fo $f$).

If $\|fx-fy\| \leqslant \alpha \|x-y\| + \beta (\|x-fx\|+\|y-fp\|) + \gamma (\|x-fy\|+\|y-fx\|)$ for al $x, y \in X$, $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + 2\beta + 2\gamma \leqslant 1$, then $f$ is called a generalized nonexpansive mapping.

A Banach sprce $X$ is said to satisfy Opial's condition if the following holds. If a sequence $\{x_n\}$ converges weakly to $x_0 \in X$ and if $x \neq x_0$ then

$$\liminf_{n\to\infty} \|x_n-x\| > \liminf_{n\to\infty} \|x_n-x_0\| .$$

Hilbert spaces and $l^p$ $(1 \leqslant p < \infty)$ spaces satisfy Opial's condition.

A mapping $I-f$ is semiclosed if $x_n \to x$ weakly and $(I-f)x_n$ converges to $y$ strongly imply that $y=(I-f)x$.

A subset $C$ of a normed linear space $X$ is said to be strashaped with respect to a poiat $p \in C$ if for all $x \in C$ and all $\lambda, 0 \leqslant \lambda \leqslant 1$, $\lambda x+(1-\lambda)\, p \in C$.

If $X$ is a normed linear space, $B$ is a subset of $X$ and $x \in X$, then

$$d(x,B)=\inf \{ \|x-y\| : y \in B \}$$

denotes the distance of $x$ to the set $B$

The set of all best approximations to $x$ is denoted by

$$D(x, B) = \{y \in B : \|x-y\|=d(x, B)\} \qquad [3].$$

Now we give our results.

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**Theorem 1.** Let $C$ be a subset of a Banach space $X$ and $T : X \to X$ be a mapping. . Let $T(\partial C) \subset C$ ( $\partial C$ stands for the boundary of $C$ ) . Let $Tx_0 = x_0$, $x_0 \in X$ and $T$ satisfy the following condition : $\|Tx - Ty\| \leqslant \alpha \|x - y\| + \beta ( \|x - Tx\| + \|y - Ty\| ) + \gamma ( \|x - Ty\| + \|y - Tx\| )$ for all x, $y \in X$, where $\alpha, \beta$ and $\gamma$ are reals with $\alpha + 2\beta + 2\gamma \leqslant 1$ .

Further, if the set $D$, the set of best $C$-approximants to $x_0$, is nonempty, closed, convex $T(D)$ compact and $T$ is continuous on $D$ then $T$ has a fixed point in $D$ closest to $x_0$ .

**Remarks.** In [1] a similar result is proved for a linear. nonexpansive map $T$ from $C$ to itself, and an extended version in [10] where $T : C \to C$ is taken, Here we do not reqire linearity, nonexpansiveness of $T$ on $X$ and $T: C \to C$ not needed only $T(\partial C) \subset C$ serves the purpose. We relax compactness of $C$ .

The following theorem (A) will be used in our proof.

Theorem A. If $f : C \to C$ is nonexpansive, where $C$ is a closed, convex subset of a normed linear space $X$ and $\overline{f(C)}$ is compact, then $f$ has a fixed point [4].

Note : Theorem A holds in $C$ is starshaped .

**Proof.** First we show that

$$\|Tx - Tx_0\| \leqslant \|x - x_0\| \quad \text{for all } x \in X.$$

Clearly,

$$\|Tx - Tx_0\| \leqslant \alpha \|x - x_0\| + \beta ( \|x - Tx\| + \|x_0 - Tx_0\| ) + \gamma ( \|x - Tx_0\| + \|x_0 - Tx\| ) .$$

Since $Tx_0 = x_0$, we get

$$\|Tx - Tx_0\| \leqslant \alpha \|x - x_0\| + \beta \|x - Tx\| + \gamma ( \|x - x_0\| + \|Tx_0 - Tx\| ) .$$

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Now,

$$\| x-Tx \| \leqslant \| x-x_0 \| + \| Tx-Tx_0 \| .$$

therefore we have

$$\| Tx-Tx_0 \| \leqslant \frac{\alpha+\beta+\gamma}{1-\beta-\gamma} \| x-x_0 \| ,$$

*i. e.* $\| Tx-Tx_0 \| \leqslant \| x-x_0 \|$ for all $x \in X$.

We claim that $T : D \to D$.

Let $y \in D$. Then

$$\| Ty-x_0 \| = \| Ty-Tx_0 \| \leqslant \| y-x_0 \| \text{ by (1)} .$$

Since y is nearest to $x_0$ so $Ty$ is also nearest to $x_0$, *i.e.* $Ty \in D$ .

Since $D$ is a nonempty closed, convex, $\overline{T(D)}$ compact subset of $X$ and $T : D \to D$ is a continuous map therefore $T$ has a fixed point by Schauder Fixed Point Theorem. So $Tx = x$ *say*, where $x \in D$ closest to $x_0$.

Corollary, In case we have a quasinonexpansive map in place of generalized nonexpansive map, the result still remains true .

Note. The result due to Carbone [2] follows as corollary. Here the condition of compactness is relaxed .

**Theorem 2.** Let $T : X \to X$ be a mapping on a Banach spaee $X$ satisfying the following :

I ) $Tx_0 = x_0$, $x_0 \in X$ .

II ) $T \{\partial C\} \subset C$, where $C$ is a subset of $X$ .

III ) The set $D$ : the set of best $C$-approximants to $x_0$ is closed, starshaped, and $F(D)$ is compact.

IV ) $T$ is nonexpansive on $D$ and $\| Tx-Tx_0 \| \leqslant \| x-x_0 \|$ for all $x \in D$.

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Then $T$ has a fixed point in $D$ closest to $x_0$.

Our proof follows similar lines as in [10] .

**Proof.** As in Theorem 1 it is easy to show that $T : D \to D$ .

Since $T$ is a nonexpansive mapping on $D$ and $D$ is starshaped with star center $P$ therefore we define

$$T_{k_n} x = k_n Tx + (1-k_n)\, p \text{ for all } x \in D.$$

where $\{k_n\}$ is a sequence of reals, $0 \leqslant k_n < 1$, converges to 1 as $n \to \infty$.

Now, $T_n : D \to D$ for each $n$, since so is $T$ ( we write $n$ in place of $k_n$ ).

Also, $\| T_n x - T_n y \| = k_n \| Tx - Ty \| \leqslant k_n \| x - y \|$, since $T$ is nonexpansive on $D$. Each $T_n$ is a contraction map with $k_n$ cantraction constant. Then each $T_n$ has a unique fixed point say $y_n$, for $n = 1, 2, \ldots$, *i. e.*

$$T_n y_n = y_n, \; n = 1, 2, \ldots$$

Since $\overline{T(D)}$ is compact therefore $\{y_n\}$ has a subseqnence $\{y_{n_i}\}$ converging to $y$ say .

We claim that $Ty = y$ in $D$ .

Since

$$y_{n_i} = T_{n_i} y_{n_i} = (1-k_{n_i})\, p + k_{n_i} Ty_{n_i} ,$$

taking limit as $i \to \infty$ and using the fact that $T$ is continuous we get that

$$y = Ty \text{ since } k_{n_i} \to 1 .$$

**Theorem 3.** Let $X$ be a Banach space with Opial's condition and $T : X \to X$ be a map. Let $Tx_0 = x_0$, $x_0 \in X$, $T(\partial C) \subset C$ where $C$ is a nonempty, weakly compact, starshaped subset of $C$ and $T$ is nonexpansive on $DU\{x_0\}$ . Then $T$ has a fixed point in $D$ closest to $x_0$ .

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Proof. It is easy to show that $T : D \to D$.

Now, define

$$f_n x = k_n fx + (1-k_n) p.$$

In this case each $f_n$ is contraction with constant $k_n < 1$, so we have $f_n x_n = x_n$ for each $n$ by Banach contraction principle.

Since $f : D \to D$ so is $f_n : D \to D$. As $D$ is weakly compact we get that $\{x_n\}$ has a subseqence $\{x_{n_i}\}$ converging weakly to $y$ say $x_{n_i} \to y$.

$$x_{n_i} - fx_{n_i} = (1-k_{n_i}) p + fx_{n_i} (1-k_{n_i}),$$

so

$$x_{n_i} - fx_{n_i} \to 0 \text{ as } i \to \infty \quad \text{since } k_{n_i} \to 1.$$

Since $X$ satisfies Opial's condition so $I-f$ is demiclosed ( *if* $x_n \to y$ and $(I-f) x_n \to z$ then $y \in D$ and $(I-f) y = z$ ).

Thus $(I-f) y = 0$ and $fy = y$.

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# ON THE INTEGRATION OF THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS ASSOCIATED WITH THE TRIPLE HYPERGEOMETRIC FUNCTION $F_D^{(3)}$

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*( Received : January 13, 1988 )*

## ABSTRACT

Interest in the Lauricella function of the fourth kind in several variables, $F_D^{(n)}$ has recently centred upon its $SL$-symmetry (Miller [5]). The cases where $n=1$ and $n=2$ have been well explored, but for higher values of $n$, no explicit study has been carried out. Fundamental systems of solutions are obtained in this paper for all the singular points of the associated partial differential system. These results are expressed entirely in terms of the functions $F_D^{(3)}$, $C_3^{(1)}$ and $C_3^{(2)}$, the functions $D_3^{1,2}$ and $D_3^{2,1}$ not being involved. This is important from the point of view of Lie theory as applied to the differtntial system under consideration in that no hypergeometric functions with restricted parameters, such as $D_3^{1,2}$, are required.

## 1. INTRODUCTION

In the integration of the partial differential system associated with the Lauricella function $F_D^{(n)}$ by the use of Pochhammer integrals,

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solutions in terms of this function and the functions $C_n^{(k)}$, all with $n+2$ parameters, are obtained. Furthermore, when $n \geqslant 3$, additional solutions present themselves, the so-colled $D_n^{p,q}$ functions, which possess $n+3$ parameters with one restriction. It has hitherto tacitly assumed that all the above types of functions would be necessary in constructing fundamental sets of integrals relative to all the singular points of the partial differential system under consideration ( Exton [1, p. 106] ) . It will be shown below that the functions with restricted parameters need not necessarily be included in such sets of integrals .

The importance of this arises from the treatment of the problem of integrating the system by means of Lie theory. The function $F_D^{(n)}$ possesses $SL$-symmetry which has aroused some interest of late ( Miller [5] ), and any solutions in the form of functions with restricted parametrs, the $D_n^{p,q}$ functions in this case, present difficulties in the application of Lie theory in this context . (Hrabowski [2] and [3])). From tne results of this study, this difficulty is removed when $n=3$, and it is conjectured that the same applies for higher values of $n$ also .

The derivation is based upon a combination of the use of series manipulation ( Olsson [6] ) and upon the use of the general properties of certain of the solutions of the system of partial differential equations under consideration ( Olsson [7] ) . This system was first formed by Lauricella [4] ( see also Srivastava and Karlsson [ 10, p. 293 *et seq* . ] ) :

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$$x_1(1-x_1)\frac{\partial^2 F}{\partial x_1^2} + x_2(1-x_1)\frac{\partial^2 F}{\partial x_1 \partial x_2} + x_3(1-x_1)\frac{\partial^2 F}{\partial x_1 \partial x_3}$$

$$+ [c-(a+b_1+1)x_1]\frac{\partial F}{\partial x_1} - b_1 x_2 \frac{\partial F}{\partial x_2} - b_1 x_3 \frac{\partial F}{\partial x_3} - ab_1 F = 0,$$

$$x_2(1-x_2)\frac{\partial^2 F}{\partial x_2^2} + x_1(1-x_2)\frac{\partial^2 F}{\partial x_2 \partial x_1} + x_3(1-x_2)\frac{\partial^2 F}{\partial x_2 \partial x_3}$$

$$+ [c-(a+b_2+1)x_2]\frac{\partial F}{\partial x_2} - b_2 x_1 \frac{\partial F}{\partial x_1} - b_2 x_3 \frac{\partial F}{\partial x_3} - ab_2 F = 0,$$

$$x_3(1-x_3)\frac{\partial^2 F}{\partial x_3^2} + x_1(1-x_3)\frac{\partial^2 F}{\partial x_3 \partial x_1} + x_2(1-x_3)\frac{\partial^2 F}{\partial x_3 \partial x_2}$$

$$+ [c-(a+b_3+1)x_3]\frac{\partial F}{\partial x_3} - b_3 x_1 \frac{\partial F}{\partial x_1} - b_3 x_2 \frac{\partial F}{\partial x_2} - ab_3 F = 0. \quad (1.1)$$

Lauricella [4] has also shown that the general integral of (1.1) may be presented as a linear combination of four of its independent integrals.

The functions $F_D^{(n)}$ and $C_n^{(k)}$ are given by

$$F_D^{(n)}(a, b_1, b_2, \ldots, b_n ; c ; x_1, x_2, \ldots, x_n)$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a, m_1+\ldots+m_n)(b_1, m_1)\ldots(b_n, m_n)}{(c, m_1+\ldots+m_n)\, m_1! \ldots m_n!} x_1^{m_1} \ldots x_n^{m_n}, \quad (1.2)$$

(Lauricella [4]),

$$C_n^{(k)}(b_1, b_2, \ldots, b_n, n, a' ; x_1, x_2, \ldots, x_n)$$

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$$= \sum_{m_1, \ldots, m_n=0}^{\infty} (b_1, m_1) \ldots (b_n, m_n)\, (a, m_{k+1} + \ldots + m_n - m_1 - \ldots - m_k)$$

$$(a', m_1 + \ldots + m_k - m_{k+1} - \ldots - m_n)\, \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!} \qquad (1.3)$$

( Exton [ 1, p. 104 ] ) .

Both series converge if $| x_j | < 1$, $j = 1, 2, \ldots, n$,

The Pochhammer symbol $( a, n )$ is given by

$$( a, n ) = a ( a+1 ) \ldots ( a+n-1 ) = \Gamma ( a+n ) / \Gamma(a) ; (a, 0) = 1.$$

The singular manifolds of the partial differential system (1.1) are easily seen to be

$x_j = 0, 1, \infty$ and $x_j = x_i$, $i, j = 1, 2, \ldots, n$, $i \neq j$ .

The singularities are formed by the intersection of two or more of the singular manifolds .

## 2. THE USE OF SERIES MANIPULATION

The triple series representation of $F_D^{(3)}$ may be written as a double series of Gauss functions :

$$F_D^{(3)} ( a, b_1, b_2, b_3 ; c ; x_1, x_2, x_3 )$$

$$= \sum_{m_1, m_2=0}^{\infty} \frac{(a, m_1 + m_2)\, (b_1, m_1)\, (b_2, m_2)}{(c, m_1 + m_2)\, m_1!\, m_2!}\, x_1^{m_1} x_2^{m_2}\, {}_2F_1 \left( \begin{matrix} a + m_1 + m_2, b_3 \\ c + m_1 + m_2 \end{matrix} ; x_3 \right). \qquad (3.1)$$

The analytic continuation of the ${}_2F_1$ series near the point at infinity is

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$$ {}_2F_1\,(a, b; c; x) = \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(b)\,\Gamma(c-a)}\,(-x)^{-a}\,{}_2F_1\,(a, a-c+1\,;\, a-b+1\,;\, 1/x) $$

$$ + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(a)\,\Gamma(c-b)}\,(-x)^{-b}\,{}_2F_1\,(b, b-c+1\,;\, b-a+1\,;\, 1/x)\,. \qquad (2.2) $$

$$ |\,arg\,(-x)\,| < \pi\,. $$

( See Whittaker and Watson [11, p. 289] ).

The terms on the right of (2.2) may be inserted into (2.1) in order to construct new solutions of (1.1). Omitting the algebra, we see that the first term yields the solution

$$ x_3^{-a}\,F_D^{(3)}\,(a, b_1, b_2, a-c+1;\, a-b_3+1;\, x_1/x_3,\, x_2/x_3,\, 1/x_3\,. \qquad (2.3) $$

If the above process is repeated, but using the second term on the right of (2.2), we obtain the solution

$$ x_3^{1-c}\quad C_3^{(2)}\,(b_1, b_2, a-c+1, b_3-c+1, c-1;\, -x_1/x_3, -x_2/x_3, -x_3)\,. \qquad (2.4) $$

Further application of similar technique gives the additional solutions

$$ x_3^{-b_3}\quad C_3^{(2)}\,(b_1, b_2, b_3, b_3-c+1, a-b_3;\, x_1/x_3,\, x_2/x_3, -1/x_3), \qquad (2.5) $$

$$ x_2^{-b_2}\,x_3^{-b_3}\quad C_3^{(1)}\,(b_1, b_2, b_3, 1+b_2+b_3-c,\, a-b_2-b_3;\, -x_1, -1/x_2, -1/x_3)\,, \qquad (2.6) $$

$$ x_2^{b_3-a}\,x_3^{-b_3}\quad C_3^{(2)}\,(b_1, 1+a-c, b_3, b_2+b_3-a,\, a-b_3;\, -x_1/x_2, -1/x_2, -x_2/x_3), \qquad (2.7) $$

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$$x_1^{-b_1}\ x_2^{-b_2}\ x_3^{-b_3}\ F_D^{(3)}(b_1+b_2+b_3-c+1,\ b_1,\ b_2,\ b_3;\ b_1+b_2+b_3-a+1;$$

$$1/x_1,\ 1/x_2\ \ 1/x_3) \qquad (2.8)$$

and $F_D^{(3)}(a, b_1, b_2, b_3;\ 1+a+b_1+b_2+b_3-c;\ 1-x_1,\ 1-x_2,\ 1-x_3$ . (2.9)

Other solutions might be obtained in like manner, but it is more convenient in the present context to make use of the general properties of the solutions of the partial differential system (1.1).

## 3 GENERAL PROPERTIES OF THE SYSTEM ASSOCIATED WITH $F_D^{(3)}$

It will be seen by inspection of the right-hand member of (1.2) that the function $F_D^{(3)}$ is unchanged if, for example, $(x_1, b_1)$ and $(x_3, b_3)$ are interchanged. Hence it follows that changes of this type may be freely applied to the system (1.1), so that, if $Z(a\ b_1, b_2, b_3;\ c;\ x_1, x_2\ x_3)$ is a solution of (2.1), then so also are $Z(a, b_3, b_2, b_1;\ c;\ x_3,\ x_2\ \ x_1)$ etc. Similarly, from the fact that (2.3), (2.8) and (2.9) are solutions of (1.1) of the form $F_D^{(3)}$ we may say that $x_3^{-a} Z(a, b_1, b_2,\ a-c+1;\ a-b_3+1;\ x_1/x_3,$ $x_2/x_3, 1/x_3)$, $x_1^{-b_1}\ x_2^{-b_2}\ x_3^{b_3}\ Z(b_1+b_2+b_3-c+1,\ b_1\ \ b_2,\ b_3;\ b_1+b_2+b_3$ $-a+1;\ 1/x_1,\ 1/x_2,\ 1/x_3)$ and $Z(a,\ b_1,\ b_2\ \ b_3\ ;\ 1+a+b_1+b_2+b_3-c;\ 1-x_1,$ $1-x_2,\ 1-x_3\ )$ are also solutions of the system (1.1). Whether or not any solutions deduced in this fashion are independent from any others must be checked separately in each case.

This technique has been employed very effectively by Olasson [7] in his investigation of the partial differential system associated with the Appell function $F_2$. By way of illustration we obtain the solution

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$$x_1^{-b_1}\, x_2^{-b_2}\, x_3^{b_1+b_2-c+1}\, F_D^{(3)}\,(b_1+b_2+b_3-c+1,\ b_1,\ b_2,\ a-c+1\ ;\ 2+b_1+b_2-c;\ x_3/x_1,\ x_3/x_2,\ x_3) \qquad (3.1)$$

by the application of (2.8) to (2.3). Another type of general solution may now be written down :

$$x_1^{-b_1}\, x_2^{-b_2}\, x_3^{b_1+b_2-c+1}\, Z(b_1+b_2+b_3-c+1,\ b_1,\ b_2,\ a-c+1;\ 2+b_1+b_2-c;\ x_3/x_1,\ x_3/x_2,\ x_3)\ . \qquad (3.2)$$

## 4. FUNDAMENTAL SYSTEMS OF SOLUTIONS OF (1.1)

By employing the results given above, it is possible to write down fundamental sets of integrals of (1.1) valid near all of the singular points of the system. It might be observed that the only singularities arising from the intersection of three singular manifolds are of the type $(\infty,1,0)$ and, as such, a complete fundamental system might be constructed, each member of which is a series convergent in the whole neighbourhood of the point in question. Otherwise, such points as $(0,1,1)$ and $(0,0,0)$ for example, are formed from the intersection of four and six singular manifolds respectively. Hence, we would expect that some of the members of each fundamental system would possess series representation which are convergent in only a part of the neighbourhood concerned. The remaining parts of each neighbourhood may be dealt with by interchanging the subscripts as indicated in Section 3. We now record the following fundamental systems, each valid near the singularity indicated with any restrictions as are appropriate ( see also Srivastava [8] and [9] ) :

(0,0,0)

$$F_D^{(3)}\,(a, b_1, b_2, b_3;\ c;\ x_1, x_2, x_3\ ),$$

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$x_1^{-b_1}\ x_2^{-b_2}\ x_3^{b_1+b_2-c+1}\ F_D^{(3)}\ (\ b_1+b_2+b_3-c+1, b_1, b_2, a-c+1\ ;\ b_1+b_2-c;\ x_3/x_1,\ x_3/x_2,\ x_2\ ),$

$x_1^{-b_1}\ x_2^{-b_2}\ C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, b_1+b_2-c+1;\ x_3/x_1,\ x_3/x_2, -x_3\ ),$

$x_1^{-b_1}\ x_2^{b_1-c+1}\ C_3^{(2)}\ (\ b_1, 1+a-c, b_3, c-b_1-1, b_1+b_2-c+1;\ -x_2/x_1, -x_2, -x_3/x_2\ ).$

$|x_1| > |x_2|,\ |x_2| > |x_3|,\ |x_1| > |x_3|.$

(!∞,0,0 )

$x_1^{-b_1}\ x_2^{-b_2}\ x_3^{b_1+b_2-c+1}\ F_D^{(3)}(\ b_1+b_2+b_2-c+1,\ b_1, b_2, a-c+1;\ b_1+b_2-c;\ x_3/x_1,\ x_3/x_2,\ x_3\ ),$

$x_1^{-b_1}\ x_2^{-b_2}\ C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, b_1+b_2-c+1;\ x_3/x_1,\ x_3/x_2, -x_3),$

$x_1^{-b_1}\ C_3^{(1)}(\ b_1, b_2, b_3, a-b_1, 1+b_1-c;\ -1/x_1, -x_2, -x_3\ ),$

$x_1^{-b_1}\ x_2^{b_1-c+1}\ C_3^{(1)}\ (\ b_1, 1+a-c, b_3\ c-b_1-1,\ b_1+b_2-c+1;\ -x_2/x_1, -x_2, -x_3/x_2\ ).$

$|x_2| > |x_3|.$

( 0,∞,∞ )

$x_1^{-a}\ F_D^{(3)}\ (\ a, b_1, b_2, a-c+1;\ a-b_3+1;\ x_1/x_3,\ x_2/x_3,\ 1/x_3\ ),$

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$$x_3^{-b_3}\ C_3^{(2)}\ (b_1,\ b_2,\ b_3,\ b_3-c+1,\ a-b_3;\ x_1/x_3,\ x_2/x_3,-1/x_3),$$

$$x_2^{-b_2}\ x_3^{-b_3}\ C_3^{(1)}\ (b_1,\ b_2,\ b_3,\ 1+b_2+b_3-c,\ a-b_2-b_3;\ -x_1,-1/x_2,\ -1/x_3)\ ,$$

$$x_2^{b_3-a}\ x_3^{-b_3}\ C_2^{(2)}\ (b_1,1+a-c,\ b_3,b_2+b_3-a,\ a-b_3;\ -x_1/x_2,\ -1/x_2,\ -x_2/x_3),$$

( $\infty,\infty,\infty$ )

$$x_3^{-a}\ F_D^{(3)}\ (\ a,\ b_1,\ b_2,\ a-c+1;\ a-b_3+1;\ x_1/x_3,\ x_2/x_3,\ 1/x_3\ ),$$

$$x_1^{-b_1}\ x_2^{-b_2}\ x_3^{-b_3}\ F_D^{(3)}\ (\ b_1+b_2+b_3-c+1,\ b_1,\ b_2,\ b_3;\ b_1+b_2+b_3-a+1;$$

$$1/x_1,\ 1/x_2,\ 1/x_3\ ),$$

$$x_3^{-b_3}\ C_3^{(2)}\ (\ b_1,\ b_2,\ b_3,\ b_3-c+1,\ a-b_3\ ;\ x_1/x_3,\ x_2/x_3,\ -1/x_3\ ),$$

$$x_2^{b_3-a}\ x_3^{-b_3}\ C_3^{(2)}\ (\ b_1,\ 1+a-c,\ b_3,\ b_2+b_3-a,\ a-b_3;$$

$$x_1/x_2,\ -1/x_2,\ x_2/x_3)\ .$$

$$|x_3|>|x_1|,\ |x_3|>|x_2|,\ |x_2|\geq|x_1|\ .$$

( $\infty,1,0$ )

$$x_1^{-b_1}\ x_2^{-b_2}\ x_3^{b_1+b_2-c+1}\ F_D^{(3)}\ (\ b_1+b_2+b_3-c+1,\ b_1,\ b_2,\ a-c+1\ ;$$

$$b_1+b_2-c;\ x_3/x_1,\ x_3/x_2,\ x_3\ ),$$

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$$x_1^{-a} F_D^{(3)}(a, b_3, b_2, a-c+1; a-b_1+1; x_3/x_1, x_2/x_1, 1/x_2).$$

$$x_1^{-b_1} x_2^{-b_2} C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, b_1+b_2-c+1; x_3/x_1, x_3/x_2, -x_3),$$

$$x_3^{-b_1} C_3^{(2)}(b_3, b_2, b_1, b_1-c+1, a-b_1; x_3/x_1, -x_2/x_1, -1/x_1).$$

$(\underline{\infty, 1, 1})$

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} (1-x_3)^{-b_3} F_D^{(3)}(c-a, b_1, a_2, c-b_1-b_2-b_3; c-a-b_3-1;$$
$$(1-x_3)/(1-x_1), (1-x_3)/(1-x_2), 1-x_3),$$

$$(1-x_1)^{-b_1} (1-x_2)^{-b_2} C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, c-a; (1-x_3)/(1-x_1),$$
$$(1-x_3)/(1-x_2), x_3-1),$$

$$(1-x_1)^{-b_1} C_3^{(1)}(b_1, b_2, b_3, a-b_1, c-a-b_2-b_3; 1/(x_1-1), x_2-1, x_3-1),$$

$$(1-x_1)^{-b_1} (1-x_2)^{c-a-b_2-b_3-1} C_3^{(1)}(b_1, c-b_1-b_2-b_3, b_3, a+b_2+b_3-c, c-a-b_2;$$
$$(x_2-1)/(1-x_1), x_2-1, (x_3-1)/(1-x_2)).$$

$$|1-x_2| > |1-x_3|.$$

$(\underline{1, \infty, \infty})$

$$(1-x_3)^{-a} F_D^{(3)}(a_1 b_1, b_2, c-b_1-b_2-b_3; a-b_3+1; (1-x_1)/(1-x_3),$$
$$(1-x_2)/(1-x_3), 1/(1-x_3)),$$

$$(1-x)^{-b_3} C_3^{(2)}(b_1, b_2, b_3, c-a-b_1-b_2, a-b_3; (1-x_1)/(1-x_3),$$
$$(1-x_2)/(1-x_3), -1/(1-x_3)),$$

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$(1-x_2)^{-b_2}\ (1-x_3)^{-b_3}\ C_3^{(1)}\ (b_1, b_2, b_3, c-a-b_3, a-b_2-b_3\ ;\ 1-x_1,$

$-1\ /\ (1-x_2),\ -1\ /\ (1-x_3)\ ),$

$(1-x_2)^{-b_2}\ (1-x_3)^{-b_3}\ C_3^{(2)}\ (b_1, c-b_1-b_2-b_3, b_3, b_2+b_3-a, a-b_3;$

$-(1-x_1)\ /\ (1-x_2),\ -1\ /\ (1-x_2),\ -(1-x_2)\ /\ (1-x_3)\ ).$

$|\ 1-x_2\ | > |\ 1-x_2\ |\ .$

( 0,0,1 )

$(1-x_1)^{-b_1}\ (1-x_2)^{-b_2}\ (1-x_3)^{c-a-b_3}\ F_D^{(3)}(c-a, b_1, b_2, c-b_1-b_2-b_3;$

$c-a-b_3-1;\ (1-x_3)\ /\ (1-x_1),\ (1-x_3)\ /\ (1-x_2),\ 1-x_3),$

$(1-x_1)^{-b_1}\ (1-x_2)^{-b_2}\ C_3^{(2)}\ (b_1, b_2, b_3, a-b_1-b_2, c-a;\ (1-x_3)\ /\ (1-x_1),$

$(1-x_3)\ /\ (1-x_2),\ x_3-1),$

$x_3^{-b_3}\ x_2^{-b_2}\ x_1^{b_2+b_3-c+1}\ F_D^{(3)}\ (b_1+b_2+b_3-c+1, b_3, b_2, a-c+1;$

$b_2+b_3-c;\ x_1\ /\ x_3,\ x_1\ /\ x_2,\ x_1\ ),$

$x_2^{-b_2}\ x_3^{-b_3}\ C_3^{(2)}\ (b_2, b_2, b_1, a-b_2-b_3, b_2+b_3-c+1;\ x_1/x_3,\ x_1/x_2,\ -x_1)\ .$

$|\ x_2\ | > |\ x_1\ |\ .$

( 1,1.0 )

$x_1^{-b_1}\ x_2^{-b_2}\ x_3^{b_1+b_2-c+1}\ F_D^{(3)}\ (1+b_1+b_2+b_3-c, b_1, b_2, 1+a-c;$

$b_1+b_2;\ x_3\ /\ x_1,\ x_3\ /\ x_2,\ x_3\ ),$

$x_1^{-b_1}\ x_2^{-b_2}\ C_3^{(2)}(b_1, b_2, b_3, a-b_1-b_2, 1+b_1+b_2+b_3-c;\ x_3/x_1,\ x_3/x_2,-x_3\ ),$

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$(1-x_1)^{c-a-b_1}\ (1-x_2)^{-b_2}\ (1-x_3)^{-b_3}\ F_D^{(3)}(c-a,\ b_3,\ b_2,\ c-b_1-b_2-b_3;$

$c-a-b_1-1;\ (1-x_1)/(1-x_3),\ (1-x_1)/(1-x_2),\ 1-x_1),$

$(1-x_2)^{-b_2}\ (1-x_3)^{-b_3}\ C_3^{(2)}(\ b_3,\ b_2,\ b_1,\ a-b_2,\ -b_3,\ c-a-b_1\ ;$

$(1-x_1)/(1-x_3),\ (1-x_1)/(1-x_2),\ x_1-1\ ).$

$|\ 1-x_2\ | > |\ 1-x_3\ |.$

( 1,1,1 )

$F_D^{(3)}(a,\ b_1\ b_2,\ b_3;\ 1+b_1+b_2+b_3+a-c;\ 1-x_1, 1-x_2,\ 1-x_3\ ),$

$(1-x_1)^{-b_1}\ (1-x_2)^{-b_2}\ (1-x_3)^{c-a-b_3}\ F_D^{(3)}(c-a,\ b_1,\ b_2,\ c-b_1-b_2-b_3;$

$c-a-b_3-1;\ (1-x_3)/(1-x_1),\ (1-x_3)/(1-x_2),\ 1-x_3),$

$(1-x_1)^{-b_1}\ (1-x_2)^{-b_2}\ C_3^{(2)}(b_1,\ b_2\ \ b_3,\ a-b_1-b_2,\ c-a-b_3;\ (1-x_3)/(1-x_1),$

$(1-x_3)/(1-x_2),\ x_3-1),$

$(1-x_1)^{-b_1}\ (1-x_2)^{c-a-b_2-b_3}\ \ C_3^{(2)}(\ b_1,\ c-b_1-b_2-b_3,\ b_3,\ a+b_2+b_3-c,\ c-a-b_3;$

$-(1-x_2)/(1-x_1),\ x_2-1,\ -(1-x_3-1)/(1-x_2)\ ).$

$|\ 1-x_1\ | > |\ 1-x_3\ |\quad |\ 1-x_2\ | > |\ 1-x_3\ |\quad |\ 1-x_1\ | > |\ 1-x_2\ |$

The remaiming singularities of thesystem may be dealt with by interchanging the subscripts in the previous results as appropriate. Fundamental sets of solutions involving convergent power series involving powers of $x_i - x_j$, $i, j = 1, 2, 3, i \neq j$, may be deduced by the

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use of the following Euler transformations of $F_D^{(3)}$ or any of their combinations using the arguments os Section 3.

$$F_D^{(3)}(a, b_1, b_2, b_3; c; x_1, x_2, x_3)$$

$$= (1x_1)^{-b_1} \quad (1-x_2)^{-b_2} \quad (1-x_3)^{-b_3} \quad F_D^{(3)}(c-a, b_1, b_2, b_3; c;$$

$$x_1/(x_1-1), x_2/(x_2-1), \quad x_3/(x_3-1))$$

$$=(1-x_1)^{-a} \quad F_D^{(3)}(a, c-b_1-b_2-b_3, b_2, b_3; c; x_1/(x_1-1), (x_1-x_2)/(x_1-1),$$

$$(x_1-x_3)/(x_1-1))$$

$$= (1-x_2)^{-a} \quad F_D^{(3)}(a, b_1, c-b_1-b_2-b_3, b_3; c; (x_2-x_1)/(x_2-1), \quad x_2/(x_2-1),$$

$$(x_2-x_3)/(x_2-1))$$

$$=(1-x_3)^{-a} \quad F_3^{(2)}(a, b_1, b_2, c-b_1-b_2-b_3; c; (x_3-x_1)/(x_3-1), (x_3-x_2)/(x_3-1),$$

$$x_3/(x_3-1)). \qquad (4.1)$$

(See Lauricella [4]). Series involving $x_i - x_j$ may be useful in covering those cases where $x_i \sim x_j$ which might otherwise be intractable. as for example, the series

$$x_1^{-a} F_D^{(2)}(a, b_1, b_2, a-c+1; a-b_3+1; x_1/x_3, x_2/x_3, 1/x_3) \qquad (4.2)$$

when $x_2 \sim x_3$. (See Olsson [6]).

It will be seen that complete fundamental systems of integrals applicable to all the singular points of the system of partial differential equations associated with $F_D^{(3)}$ may be constructed entirely from

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the functions $F_D^{(3)}$, $C_3^{(1)}$ and $C_3^{(2)}$ without having recourse to the functions with restricted parameters $D_3^{1,2}$ and $D_3^{2,1}$. It is conjectured that the analogous situation applies in the case of the system $F_D^{(n)}$ for all values of $n$.

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# KARLSSON'S MULTIPLE HYPERGEOMETRIC FUNCTION AND ITS CONFLUENT FORMS

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## ABSTRACT

In the present paper, We derive confluent forms of Karlsson's multiple hypergometric function and obtain their generating relations and integral representations with their applications in obtaining their recurrence relations .

## 1. INTRODUCTION

Recently, Chandel and Gupta [ 1 ] introduced intermediate Lauricella's multiple hypergeometric functions ${}^{(k)}F_{AC}^{(n)}$ , ${}^{(k)}F_{AD}^{(n)}$ , ${}^{(k)}F_{BD}^{(n)}$ and their confluent forms . Subsequently, Karlsson [4] also introduced one more possible intermediate Lauricella function ${}^{(k)}F_{CD}^{(n)}$ defined by

$$(1.1)\quad {}^{(k)}F_{CD}^{(n)}[a, b, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n] .$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b)_{m_{k+1}+\ldots+m_n} (b_1)_{m_1} \ldots (b_k)_{m_k}}{(c)_{m_1+\ldots+m_k} (c_{k+1})_{m_{k+1}} \ldots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!} ,$$

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which for $k=0$ reduces to Lauricella's function $F_C^{(n)}$ and for $k=n$ reduces to Lauricella's function $F_D^{(n)}$ [5].

The purpose of this paper is to study the confluent hypergeometric functions of the fourth intermediate Lauricella's function of Karlsson [4]. We shall discuss special cases, generating relations, and integral representations with their applications in obtaining recurrence rhlations of these multiple hypergeometric functions.

## 2. CONFLUENT FORMS

For confluent forms, we consider

$$(2\,1)\quad \lim_{b_1,\ \ldots,\ b_k\to\infty} {}^{(k)}F_{C_D}^{(n)}[a,\ b,\ b_1,\ \ldots,\ b_k;\ c,\ c_{k+1},\ \ldots,\ c_n;\ \frac{x_1}{b_1},\ \ldots,\ \frac{x_k}{b_k},\ x_{k+1},\ \ldots,\ x_n]$$

$$= \sum_{m_1,\ \ldots,\ m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n}\,(b)_{m_{k+1}+\ldots+m_n}}{(c)_{m_1+\ldots+m_k}\,(c_{k+1})_{m_{k+1}}\ldots(c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{C_D}[a,\ b;\ c,\ c_{k+1},\ \ldots,\ c_n;\ x_1,\ \ldots,\ x_n],\quad k\neq 0.$$

For $k=0$, it reduces to Lauricella's $F_C^{(n)}$.

$$(2.2)\quad \lim_{b\to\infty} {}^{(k)}F_{C_D}^{(n)}[a,\ b,\ b_1,\ \ldots,\ b_k;\ c,\ c_{k+1},\ \ldots,\ c_n;\ x_1,\ \ldots,\ x_k,$$

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$$\frac{x_{k+1}}{b}, \ldots, \frac{x_n}{b}]$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b_1)_{m_1} \ldots (b_k)_{m_k}}{(c)_{m_1+\ldots+m_k} (c_{k+1})_{m_{k+1}} \ldots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(2)}\phi^{(n)}_{CD} [a, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n], \quad k \neq n.$$

For $k=n$, it reduces to Lauricella's $F_D^{(n)}$,

$$(2.3) \quad \lim_{a \to \infty} {}^{(k)}F^{(n)}_{CD} [a, b, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; \frac{x_1}{a}, \ldots, \frac{x_n}{a}]$$

$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(b)_{m_{k+1}+\ldots+m_n} (b)_{m_1} \ldots (b_k)_{m_k}}{(c)_{m_1+\ldots+m_k} (c_{k+1})_{m_{k+1}} \ldots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(3)}\phi^{(n)}_{CD} [b, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n], \quad k \neq 0, k \neq n.$$

For $k=n$ it reduces to $\phi^{(n)}_{(2)}$ while for $k=0$ it reduces to $\Psi_2^{(n)}$.

$$(2.4) \quad \lim_{c_{k+1}, \ldots, c_n \to 0} {}^{(k)}F^{(n)}_{CD} [a, b, b_1, \ldots, b_k; c, c_{k+1}, \ldots c_n; x_1, \ldots, x_k, c_{k+1} x_{k+1}, \ldots, c_n x_n]$$

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$$= \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\ldots+m_n} (b)_{m_{k+1}+\ldots+m_n} (b_1)_{m_1} \ldots (b_k)_{m_k}}{(c)_{m_1+\ldots+m_k}} \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

$$= {}^{(k)}_{(4)}\phi^{(n)}_{CD} [a, b, b_1, \ldots, b_k ; c; x_1, \ldots, x_n] \ . \ k \neq n \ .$$

For $k=n$, it reduces to Lauricella's $F_D^{(n)}$.

## 3. INTERGRAL REPRESENTATIONS

Making an appeal to Srivastava [ 6, p. 101 ]

$$(3.1) \quad (\lambda, m) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-t} \ t^{\lambda+m-1} \ dt,$$

and Erdélyi [ 2, p. 13 ]

$$(3.2) \quad \frac{1}{(\lambda, m)} = \frac{\Gamma(\lambda)}{2\pi i} \int_{-\infty}^{(0+)} e^{t} \ t^{-\lambda-m} \ dt,$$

$\mathrm{Re}\,(\lambda) > 0, \ m = 0, 1, 2, \ldots,$

we derive the following integral representations :

$$(3.3) \quad {}^{(k)}F^{(n)}_{CD} [a, b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_n]$$

$$= \frac{1}{2\pi i} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^{(0+)} e^{-(s+t-u)} \ s^{a-1} \ t^{b-1} \ u^{-c} (1-sx_1/u)^{-b_1}$$

$$\ldots (1-sx_k/u)^{-b_k} \ {}_0F_1 [-; c_{k+1}; stx_{k+1}] \ldots {}_0F_1 [-; c_n; stx_n] \ ds \ dt \ du,$$

where $\mathrm{Re}\,(a) > 0$, $\mathrm{Re}\,(b) > 0$.

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$$(3.4)\quad {}^{(k)}_{(1)}\phi^{(n)}_{CD}\,[\,a,\ b;\ c, c_{k+1}, \ldots, c_n\ ;\ x_1, \ldots, x_n\,]$$

$$= \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \int_{-\infty}^{(0+)} e^{-(s+t-u)}\, e^{s/u(x_1+\ldots+x_k)}$$

$$s^{a-1}\ t^{b-1}\ u^{-c}$$

$$\cdot\ {}_0F_1\,[\,-\ ;\ c_{k+1}\ ;\ st\ x_{k+1}\,]\ \ldots\ {}_0F_1\,[\,-\ ;\ c_n\ ;\ stx_n\,]\ ds\ dt\ du,$$

where Re $(a) > 0$, Re $(b) > 0$,

$$(3.5)\quad {}^{(k)}_{(2)}\phi^{(n)}_{CD}\,[\,a,\ b_1, \ldots, b_k\ ;\ c,\ c_{k+1}, \ldots, c_n\ ;\ x_1, \ldots, x_n\,]$$

$$= \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a)} \int_0^\infty \int_{-\infty}^{(0+)} e^{-(s-u)}\, s^{a-1}\ u^{-c}\ (1-sx_1/u)^{-b_1}\ \ldots\ (1-sx_k/u)^{-b_k}$$

$$\cdot\ {}_0F_1\,[\,-\ ;\ c_{k+1}\ ;\ sx_{k+1}\,]\ \ldots\ {}_0F_1\,[\,-\ ;\ c_n\ ;\ sx_n\,]\ ds\ du,$$

where Re $(a) > 0$.

$$(3.6)\quad {}^{(k)}_{(3)}\phi^{(n)}_{CD}\,[\,b,\ b_1, \ldots, b_k\ ;\ c,\ c_{k+1}, \ldots, c_n\ ;\ x_1, \ldots, x_n\,]$$

$$= \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(b)} \int_0^\infty \int_{-\infty}^{(0+)} e^{-(t-u)}\, t^{b-1}\ (1-x_1/u)^{-b_1}\ \ldots\ (1-x_k/u)^{-b_k}$$

$$\cdot\ {}_0F_1\,[\,-\ ;\ c_{k+1}\ ;\ tx_{k+1}\,]\ \ldots\ {}_0F_1\,[\,-\ ;\ c_n\ ;\ tx_n\,]\ dt\ du,$$

where Re $(b) > 0$.

$$(3.7)\quad {}^{(k)}_{(4)}\phi^{(n)}_{CD}\,[\,a, b,\ b_1, \ldots, b_k\ ;\ c;\ x_1, \ldots, x_n\,]$$

$$= \frac{1}{2\pi i}\,\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^\infty \int_0^\infty \int_{-\infty}^{(0+)} e^{-(s+t-u)}\, e^{st\,(x_{k+1}+\cdots+x_n)}\, s^{a-1}\ t^{b-1}$$

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$$. u^{-c} (1-x_1 s/u)^{-b_1} \ldots (1-x_k s/u)^{-b_k} \, ds\, dt\, du,$$

where Re $(a) > 0$, Re $(b) \geq 0$.

$$(3.8) \quad {}^{(k)}_{(4)}\phi^{(n)}_{CD}[a, b, b_1, \ldots, b_k ; c; x_1, \ldots, x_n]$$

$$= \frac{1}{\Gamma(a)\,\Gamma(b)\,\Gamma(b_1)\ldots\Gamma(b_k)} \int_0^\infty \ldots (k+2) \ldots \int_0^\infty e^{-s} [1 - t(x_{k+1} + \ldots + x_n - 1) + t_1 + \ldots + t_k]$$

$$. s^{a-1} t^{b-1} t_1^{b_1-1} \ldots t_k^{b_k-1} {}_0F_1 [-; c, x_1 st_1 + \ldots + x_k st_k]$$

$$ds\, dt\, dt_1 \ldots dt_k,$$

Re $(a) > 0$, Re $(b) > 0$, and Re $(b_\lambda) > 0$, $i = 1, \ldots k$,

$$(3.9) \quad {}^{(k)}F^{(n)}_{CD}[a, b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n]$$

$$= \frac{1}{\Gamma(a)\,\Gamma(b)\,\Gamma(b_1)\ldots\Gamma(b_k)} \int_0^\infty \ldots (k+2) \ldots \int_0^\infty e^{-(s+t+t_1+\ldots+t_k)}$$

$$. s^{a-1} t^{b-1} t_1^{b_1-1} \ldots t_k^{b_k-1} {}_0F_1 [-; c; st_1 x_1 + \ldots + st_k x_k]$$

$${}_0F_1 [-; c_{k+1}; st\, x_{k+1}]\, {}_0F_1 [-; c_n; st\, x_n)\, ds\, dt\, dt_1 \ldots dt_k,$$

Re $(a) > 0$, Re $(b) > 0$, Re $(b_i) > 0$, $i = 1, \ldots, k$.

$$(3.10) \quad {}^{(k)}_{(1)}\phi^{(n)}_{CD}[a, b; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_n]$$

$$= \frac{1}{\Gamma(a)\,\Gamma(b)} \int_0^\infty \int_0^\infty s^{a-1} t^{b-1} e^{-(s+t)} {}_0F_1 [-; c; sx_1 + \ldots + sx_k]$$

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$\cdot\ {}_0F_1\ [\ -\ ;\ c_{k+1}\ ;\ st_{k+1}\ x_{k+1}\ ]\ \ldots\ {}_0F_1\ [\ -\ ;\ c_n\ ;\ st_n\ x_n\ ]\ ds\ dt,$

$\mathrm{Re}\ (a) \geqslant 0,\ \mathrm{Re}\ (b) \geqslant 0\ .$

$$(3.11)\quad {}^{(k)}_{(2)}\phi^{(n)}_{CD}\ [\ a,\ b_1\ ,\ \ldots\ ,\ b_k;\ c,\ c_{k+1}\ ,\ \ldots\ c_n\ ;\ x_1,\ \ldots\ ,\ x_k\ ]$$

$$= \frac{1}{\Gamma(a)\ \Gamma(b_1)\ \ldots\ \Gamma(b_k)} \int_0^\infty \ldots (k+1) \ldots \int_0^\infty e^{-(s+t_1+\ldots+t_k)}$$

$$s^{a-1}\ t_1^{b_1-1}\ \ldots\ t_k^{b_k-1}$$

$${}_0F_1\ [\ -\ ;\ c\ ;\ x_1\ t_1\ s+\ \ldots\ +x_k\ t_k\ s\ ]\ {}_0F_1\ [\ -\ ;\ c_{k+1}\ ;\ x_{k+1}\ s\ ]$$

$$\ldots\ {}_0F_1\ [\ -;\ c_n;\ x_n s\ ]\ ds\ dt_1\ \ldots\ dt_k,$$

$\mathrm{Re}\ (a) > 0,\ \mathrm{Re}\ (b_i) > 0,\ i=1\ ,\ \ldots\ ,\ k.$

$$(3.12)\quad {}^{(k)}_{(3)}\phi^{(n)}_{CD}\ [\ b,\ b_1\ ,\ \ldots\ ,\ b_k;\ c,\ c_{k+1}\ ,\ \ldots\ ,\ c_n;\ x_1\ ,\ \ldots\ ,\ x_n\ ]$$

$$= \frac{1}{\Gamma(b)\ \Gamma(b_1)\ \ldots\ \Gamma(b_k)} \int_0^\infty \ldots (k+1) \ldots \int_0^\infty e^{-(\ t\ +\ t_1\ +\ \ldots\ +\ t_k\ )}$$

$$t^{b-1}\ t_1^{b_1-1}\ \ldots\ t_k^{b_k-1}$$

$${}_0F_1\ [\ -\ ;\ c;\ x_1\ t_1\ +\ \ldots\ +\ x_k\ t_k\ ]\ {}_0F_1\ [\ -\ ;\ c_{k+1}\ ;\ x_{k+1}\ t\ ]\ \ldots$$

$$\ldots\ {}_0F_1\ [\ -\ ;\ c_n\ ;\ x_n\ t\ ]\ dt\ dt_1\ \ldots\ dt_k,$$

$\mathrm{Re}\ (b) > 0,\ \mathrm{Re}\ (b_i) > 0,\ i=1\ \ldots\ k.$

## 4. RECURRENCE RELATIONS

Using the result due to Exton [ 3, p. 115 (3.5) ]

$$(4.1)\quad {}_0F_1\ [-;\ c-1;\ x\ ] - {}_0F_1\ [\ -\ ;\ c;\ x\ ] - \frac{x}{c(c-1)}\ {}_0F_1\ [\ -\ ;\ c+1\ ;\ x\ ] = 0$$

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we derive the following recurrence reletions from (3.3), (3.4) and (3.5), respectively :

$$(4.2)\quad {}^{(k)}_{CD}F^{(n)}\left[a, b, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$= {}^{(k)}_{CD}F^{(n)}\left[a, b, b_1, \ldots, b_k; c, c_{k+1}, \cdots, c_{k+j-1}, c_{k+j}-1, c_{k+j+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)}\ {}^{(k)}_{CD}F^{(n)}\left[a, b, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_{k+j-1}, c_{k+j}+1, c_{k+j+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

where $j=1, \ldots, n-k$.

$$(4.3)\quad {}^{(k)}_{(1)}\phi^{(n)}_{CD}\left[a, b; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{CD}\left[a, b; c, c_{k+1}, \ldots, c_{k+j-1}, c_{k+j}-1, c_{k+j+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)}\ {}^{(k)}_{(1)}\phi^{(n)}_{CD}\left[a, b; c, c_{k+1}, \ldots, c_{k+j-1}, c_{k+j}+1, c_{k+j+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

where $j=1, \ldots, n-k$.

$$(4.4)\quad {}^{(k)}_{(2)}\phi^{(n)}_{CD}\left[a, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

$$= {}^{(k)}_{(2)}\phi^{(n)}_{CD}\left[a, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_{k+j-1}, c_{k+j}-1, c_{k+j+1}, \ldots, c_n; x_1, \ldots, x_n\right]$$

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$$- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)} {}^{(k)}_{(2)}\phi^{(n)}_{CD} [\, a, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_{k+j-1}, c_{k+j}+1,$$

$$c_{k+j+1}, \ldots, c_n ; x_1, \ldots, x_n \,],$$

where $j = 1, \ldots, n-k$.

$$(4.5) \quad {}^{(k)}_{(3)}\phi^{(n)}_{CD} [\, b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_n \,]$$

$$= {}^{(k)}_{(3)}\phi^{(n)}_{CD} [\, b, b_1, \ldots, b_k ; c; c_{k+1}, \ldots, c_{k+j-1}, c_{k+j}-1,$$

$$c_{k+j+1}, \ldots, c_n ; x_1, \ldots, x_n \,]$$

$$- \frac{x_{k+j}}{c_{k+j}(c_{k+j}-1)} {}^{(k)}_{(3)}\phi^{(n)}_{CD} [\, b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_{k+j-1}, c_{k+j}+1,$$

$$c_{k+j+1}, \ldots, c_n ; x_1, \ldots, x_n \,],$$

$j = 1, \ldots, n-k$.

## 5. GENERATING RELATIONS

From the definitions of Section 2, we obtain the following generating relations :

$$(5.1) \quad (1-t)^{-a} {}^{(k)}F^{(n)}_{CD} [\, a, b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_n ;$$

$$\frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t} \,]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r t^r}{r!} {}^{(k)}F^{(n)}_{CD} [\, a+r, b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots c_n ; x_1, \ldots, x_n \,]$$

$$(5.2) \quad (1-t)^{-b} {}^{(k)}F^{(n)}_{CD} [\, a, b, b_1, \ldots, b_k ; c, c_{k+1}, \ldots, c_n ; x_1, \ldots, x_k,$$

$$\frac{x_{k+1}}{1-t}, \ldots, \frac{x_n}{1-t} \,]$$

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$$= \sum_{r=0}^{\infty} \frac{(b)_r\, t^r}{r!}\ {}^{(k)}_{CD}F^{(n)}[\, a, b+r, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1 \ldots, x_n\,],$$

$$(5.3)\quad (1-t)^{-b_i}\ {}^{(k)}_{CD}F^{(n)}[\, a, b, b_1, \ldots, b_k\,;\, c, c_{k+1}, \ldots, c_n\,;$$

$$x_1, \ldots, x_{i-1}, \frac{x_i}{1-t}\ x_{i+1}, \ldots, x_k, \ldots, x_n\,]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}_{CD}F^{(n)}\ [\, a, b, b_1, \ldots, b_{i-1}, b_i+r, b_{i+1}, \ldots, b_k\,;\, c,$$

$$c_{k+1}, \ldots, c_n\ ;\ x_1, \ldots, x_n\,],\quad i=1, \ldots, k,$$

$$(5.4)\quad (1-t)^{-a}\ {}^{(k)}_{(1)}\phi^{(n)}_{CD}[a, b; c, c_{k+1}, \ldots, c_n; \frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t}\,]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ {}^{(k)}_{(1)}\phi^{(n)}_{CD}\,[\, a+r, b\,;\, c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\,],$$

$$(5.5)\quad (1-t)^{-a}\ {}^{(k)}_{(2)}\phi^{(n)}_{CD}\ [\, a, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n\ ;$$

$$\frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t}\,]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ {}^{(k)}_{(2)}\phi^{(n)}_{CD}\,[\, a+r, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n\,;\, x_1, \ldots, x_n],$$

$$(5.6)\quad (1-t)^{-b}\ {}^{(k)}_{(3)}\phi^{(n)}_{CD}\,[\, b, b_1, \ldots, b_k\,;\, c, c_{k+1}, \ldots, c_n\,;\, x_1, \ldots, x_k,$$

$$\frac{x_{k+1}}{1-t}, \ldots, \frac{x_n}{1-t}\,]$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r\, t^r}{r!}\ {}^{(k)}_{(3)}\phi^{(n)}_{CD}\,[\, b+r, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\,],$$

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$$(5.7)\quad (1-t)^{-a}\ \underset{(4)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a, b, b_1, \ldots, b_k; c; \frac{x_1}{1-t}, \ldots, \frac{x_n}{1-t}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(a)_r\, t^r}{r!}\ \underset{(4)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a+r, b, b_1, \ldots, b_k; c; x_1, \ldots, x_n\right].$$

$$(5.8)\quad (1-t)^{-b}\ \underset{(1)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a, b; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_k, \frac{x_{k+1}}{1-t}, \ldots, \frac{x_n}{1-t}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r\, t^r}{r!}\ \underset{(1)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a, b+r; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right],$$

$$(5.9)\quad (1-t)^{-b}\ \underset{(4)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a, b, b_1, \ldots, b_k; c; x_1, \ldots, x_k, \frac{x_{k+1}}{1-t}, \ldots \frac{x_n}{1-t}\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b)_r\, t^r}{r!}\ \underset{(4)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a, b+r, b_1, \ldots b_k; c; x_1, \ldots, x_n\right],$$

$$(5.10)\quad (1-t)^{-b_i}\ \underset{(2)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a\, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_{i-1}, \frac{x_i}{1-t}, x_{i+1}, \ldots, x_k, \ldots, x_n\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ \underset{(2)}{\overset{(k)}{\phi}}{}^{(n)}_{CD}\left[a, b_1 \ldots, b_{i-1}, b_i+r, b_{i+1}, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right].$$

$i=1, \ldots, k.$

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$$(5.11)\quad (1-t)^{-b_i}\ {}^{(k)}_{(3)}\phi^{(n)}_{CD}\left[b, b_1, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_{i-1}, \frac{x_i}{t}, x_{i+1}, \ldots, x_k, \ldots, x_n\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\ {}^{(k)}_{(3)}\phi^{(n)}_{CD}\left[b, b_1, \ldots, b_{i-1}, b_i+r, b_{i+1}, \ldots, b_k; c, c_{k+1}, \ldots, c_n; x_1, \ldots, x_n\right],$$

$i=1, \ldots, k.$

$$(5.12)\quad (1-t)^{-b_i}\ {}^{(k)}_{(4)}\phi^{(n)}_{CD}\left[a, b, b_1, \ldots, b_k; c, x_1, \ldots, x_{i-1}, \frac{x_i}{1-t}, x_{i+1}, \ldots, x_k, \ldots, x_n\right]$$

$$= \sum_{r=0}^{\infty} \frac{(b_i)_r\, t^r}{r!}\left[a, b, b_1, \ldots, b_{i-1}, b_i + r, b_{i+1}, \ldots, b_k; c; x_1, \ldots, x_n\right],$$

$i=1, \ldots, k.$

In a forthcoming paper, we shall discuss fractional derivatives and Eulerian integral representations of the above functions.

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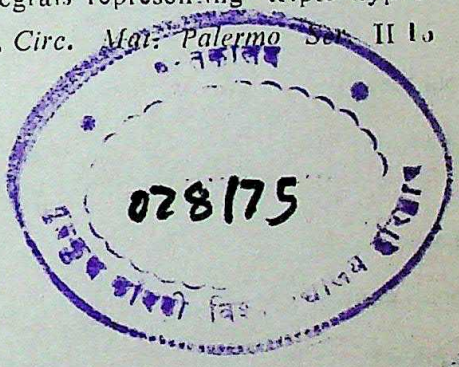


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